Singular perturbation analysis of an additive increase multiplicative decrease control algorithm under time-varying buffering delays.*

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Summary. A non-linear model for the description of packet-switched communication networks is presented. This model is combined with a model of an additive increase, multiplicative decrease end-to-end controller to obtain a global description of a controlled packet switched network with buffering delays. The global stability of this general model is then analysed using singular perturbation analysis. It is shown that under sufficiently small propagation delays, this general model is globally and exponentially stable.

1 Introduction

Since the Jacobson's seminal paper [2] describing the dynamical behaviour of the Transfert Control Protocol (TCP), this congestion control has been able to cope with an unpreceded growth of Internet users and a tremendous increase of available bandwidth.

Many papers have been published to analyse and explain this success and many disciplines have been used to study congestion control algorithms : A review of Internet congestion control may be found in [6].

The contribution of this paper is twofold : firstly, we present a global fluid flow model for the network dynamics description using the buffer levels as our state vector which is a natural way of including buffering delays.

Secondly, we combine this network model with a model of an additive increase, multiplicative decrease model studied by Kelly in [4, 3]. This model is adapted and viewed as a controller connected in feedback to our network model. The dynamics of these two systems are analysed separately and the

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singular perturbation theory is used to obtained results on the global stability of the closed loop system.

Stability results that take into account propagation delays are studied in [3, 10] where sufficient conditions for local stability are derived. In [9], the global stability of a congestion control algorithm derived from the same congestion model is studied with propagation delays and time-varying buffering delays. However, a single buffer is considered. In contrast, in this paper, the global stability of an arbitrary topology is studied.

2 A global fluid flow model for packet-switched networks

A common framework to analyse packet-switched networks is to abstract the existence of each individual packet into the concept of a continuous flow of information. This is the so-called *fluid flow modelling paradigm* and is not without comparison with fluid dynamics which does not account for each individual fluid molecules.

Such a fluid flow model might be constructed as follows (see also [1]) : Consider the Fig. 1 showing a typical network buffer. The level of data in the buffer at time t is noted $x(t)$. The buffer is fed at a rate $u(t)$ and is drained at a rate $v(t)$.

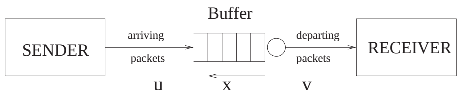

Fig. 1. A simple network with a sender, a router and a receiver

The output rate $v(t)$ may be expressed as a function of the buffer level $x(t)$ by way of a *processing rate function* noted $r(x)$. This function satisfies the following properties :

- The function $r(x)$ is $\mathcal{C}^1$
- $r(x=0)=0$: There is no output if the buffer is empty
- $\lim_{x\to\infty} r(x) = \mu > 0$: The output rate of the buffer is limited by a constant μ which is the maximum link capacity
- $r(x)$ is monotonically increasing

Note that, with these properties, the processing rate function may be rewritten as $r(x) = \tilde{r}(x)x$. With these notations, a fluid flow model for the system depicted in Fig. 1 may be written as:

$$\dot{x} = u - r(x)$$

In order, to obtain a global description of a general network, let us now consider Fig. 2 showing the interconnection between two adjacent buffers.

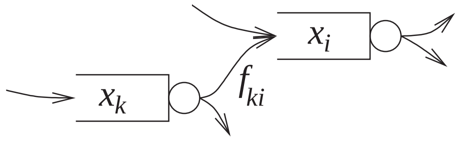

Fig. 2. Interconnection of two adjacent buffers

We assume that there are n_b buffers numbered from 1 to n_b (index set $\mathcal{I}_b$), n_s senders numbered from n_b+1 to n_b+n_s (index set $\mathcal{I}_s$) and n_r receivers numbered from n_b+n_s+1 to $n_b+n_s+n_r$ (index set $\mathcal{I}_r$). The following definitions and notations are introduced :

$\mathcal{A}_i \subset \mathcal{I}_b$ is the index set of upstream buffers connected to the buffer i;
$\mathcal{S}_i \subset \mathcal{I}_s$ is the index set of senders connected to the buffer i;
$\mathcal{B}_i \subset \mathcal{I}_b$ is the index set of downstream buffers connected to the buffer i;
$\mathcal{R}_i \subset \mathcal{I}_r$ is the index set of receivers connected to the buffer i;
$x_i(t)$ is the content (or occupancy) of the buffer i;
$v_i(t)$ is the flow of packets entering the buffer i;
$w_i(t)$ is the flow of packets leaving the buffer i;
$f_{ij}(t)$ is the flow of packets on the link $i \to j$.

As before, the flow balance equation around the buffer i is written :

$$\dot{x}_i = v_i - w_i = \sum_{k \in \mathcal{S}_i \cup \mathcal{A}_i} f_{ki} - \sum_{j \in \mathcal{B}_i \cup \mathcal{R}_i} f_{ij} \qquad i = 1, n \tag{1}$$

The flow f_{ki} between the buffer k and the buffer i is written as the product of two terms :

$$f_{ki} = \alpha_{ki} r_k(x_k)$$

where

α_{ki} is the routing variable, that is to say the fraction of the output flow from the buffer k that is going toward the buffer i. We have that $\sum_{i \in \mathcal{B}_k \cup \mathcal{R}_k} \alpha_{ki} = 1$

$r_k(x_k)$ is the processing rate function of the buffer k

Finally, the sender flow rates f_{ki} with $k \in \mathcal{I}_s$ are modelled as :

$$f_{ki} = d_k \quad k \in \mathcal{I}_s$$

with d_k, the *demand* of the sender k. With these notations and definitions, it is readily seen that the general form of the state equations for a communication network is:

$$\dot{x}_i = \sum_{\ell \in \mathcal{S}_i} d_\ell + \sum_{k \in \mathcal{A}_i} \alpha_{ki} r_k(x_k) - \sum_{j \in \mathcal{B}_i \cup \mathcal{R}_i} \alpha_{ij} r_i(x_i) \quad i \in \mathcal{I}_b \tag{2}$$

This general state space model of communication networks is a *compartmental network system* which can be written in a compact matrix form:

$$\dot{x} = G(x)x + Bd \tag{3}$$

where

x is an n-dimensional state vector with entries x_i, $i \in \mathcal{I}_b$;
d is an input vector with non-zero entries of the form d_ℓ, $\ell \in \mathcal{I}_s$;
B is a $n_b \times n_s$ permutation matrix that connects each sender to its corresponding buffer;
$G(x) = [g_{ij}(x)]$ is a so-called *compartmental matrix* with the following properties:

1. $G(x)$ is a so-called Metzler matrix with non-negative off-diagonal entries which are either 0 or of the form:

$$g_{ij}(x) = \alpha_{ji} \tilde{r}_j(x_j) \qquad i, j \in \mathcal{I}_b \;\; i \neq j$$

(note the inversion of the indexes !) ;

2. The diagonal entries are non positive and have the form:

$$g_{ii}(x) = -\tilde{r}_i(x_i) - \sum_{j \neq i} g_{ij}(x) \quad i, j \in \mathcal{I}_b \;\; k \in \mathcal{I}_r$$

3. The matrix $G(x) = [g_{ij}(x)]$ is diagonally dominant:

$$|g_{ii}(x)| \geqslant \sum_{j \neq i} g_{ji}(x)$$

An important property is that a compartmental system of the form (3) is a *non-negative* system. In the specific case of a communication network, this means that if the initial buffer loads are non-negative ($x_i(0) \geqslant 0$), then the buffer loads are guaranteed to be non-negative along the system trajectories ($x_i(t) \geqslant 0$, $\forall t$) in accordance with the physical reality.

3 Model of the AIMD mechanism

Each sender $r \in \mathcal{I}_s$, with demand d_r is connected to a distinct receiver through the network (we therefore consider the case where $n_r = n_s$). The rate of information reaching the receiver r at time t is denoted $D_r(t)$. The function $D_r(t)$ is also referred to as the *excretion* rate in the compartmental system framework. The set of buffers (network resources) that are used to connect the sender to the receiver r is called a *route*. We write $j \in r$ to indicate that a resource $j \in \mathcal{I}_b$ belongs to the route $r \in \mathcal{I}_s$.

A congestion control is a decentralised control algorithm that forces the controlled system to operate at an equilibrium point which must satisfy some global properties of fairness or should maximize the global utilization of the available resources. This is illustrated in Fig. 3 which shows a network with 2 routes and 5 network buffers. The receivers relay a congestion indication carried by data packets back to the sender that must react to alleviate the congestion. This is referred to as end-to-end (e2e) congestion control. In order

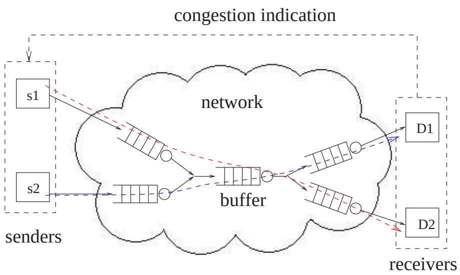

Fig. 3. Packet switched network with end-to-end congestion control.

to probe gradually for available bandwidth and to react quickly to congestion indication, congestion control law, such as TCP for instance, implement a mechanism known as *additive increase, multiplicative decrease* [2]. The time evolution of the demand of the sender r may be expressed by the following system of differential equation ([4, 3, 10]) :

$$\frac{d}{dt}d_r(t) = \kappa_r\Big(w_r - d_r(t)\sum_{j\in r}\eta_j(t)\Big) \quad r \in \mathcal{I}_s \tag{4}$$

where

$$\eta_j(t) = p_j\Big(\sum_{s:s\in j} d_s(t)\Big) \tag{5}$$

Assume that $w_r, \kappa_r > 0$ and that $p_j(x), x \geqslant 0$, is non-negative, continuous, strictly increasing function. $\eta_j(t)$ may be interpreted as a price advertised by the resource j which naturally increases with the total flow using the resource j. From the point of view of the source s_r, the aggregated price of all the resources used by the route r represents the congestion information as represented in Fig. 3. The sum

$$C_r = \sum_{j\in r}\eta_j(t) \tag{6}$$

can be interpreted as the cost associated with the route r. Equation (4) expresses that, if this cost is low, the demand increases almost linearly with w_r

whereas if the cost is high, the demand decreases almost exponentially. In the next Section, a function C_r which takes into account the particularities of the model (3) will be redefined.

It is shown in [3, 4] that the unique equilibrium point of system (4)-(5) given by

$$d_r = \frac{w_r}{\sum_{j\in r} \eta_j}$$

is globally and asymptotically stable. In the next Section, the global network model (3) is combined with the congestion control (4) to obtain a global controlled network model with buffering delays. Such an equilibrium point has been termed *proportionaly fair* by Kelly.

4 Global network model with E2E congestion control

In order to connect systems (3) and (4), a cost function $C_r(x)$ has to be defined. This function must reflect the congestion state of the route r and plays the role of eq. (6) but takes into account the buffering delays.

Consider the extension of system (3) given by :

$$\begin{cases} \dot{x} &= G(x)x + Bd \\ \dot{z}_r &= D_r - d_r \quad r \in \mathcal{I}_s \end{cases} \tag{7}$$

Suppose that $z_r(0) = z_0$, we then have that $z_0 - z_r$ is equal to the quantity of information that has been injected into the route r in transit towards the receiver . We may therefore consider a cost $C_r(-z_r)$ where C_r is a non-decreasing function with $\lim_{x\to-\infty} C_r(x) = 0$ and $\lim_{x\to+\infty} C_r(x) = +\infty$. The global model now becomes :

$$\begin{cases} \dot{x} &= G(x)x + Bd \\ \dot{z}_r &= D_r - d_r \quad r \in \mathcal{I}_s \\ \dot{d}_r &= \kappa_r \Big(w_r - d_r(t) C_r(-z_r) \Big) \quad r \in \mathcal{I}_s \end{cases} \tag{8}$$

In the following section, a singular perturbation analysis is performed to analyse the stability of system (8).

5 Singular perturbation analysis

In eq. (8), the parameters κ_r have dimension $1/[s]$. It is therefore natural to rewrite them as $\kappa_r = 1/T_r = \gamma_r/T$ where T_r corresponds to some fixed propagation delays on the route r. The parameter T then appears as an obvious choice for a singular parameter (See [5] for the development of this theory). Singular perturbation analysis therefore allows us to view system (8) as the superposition of two dynamics : fast dynamics governed by the E2E control on one hand, and network dynamics, on the other hand.

5.1 The boundary-layer system (fast dynamics)

Let us now focus on the e2e dynamics. Viewing the state variables x, z as fixed parameters, the dynamics are given by

$$T\dot{d}_r = \gamma_r(w_r - d_r C_r(-z_r)) \stackrel{\text{def}}{=} g_r(z, d)$$

This system has a single equilibrium point

$$d_r = \frac{w_r}{C_r(-z_r)} \stackrel{\text{def}}{=} h_r(z_r)$$

which can be shifted to the origin with the change of variable $y_r = d_r - h_r$. In order to reveal the two different time scales inherent to the singular perturbation analysis, the following change of variable is also performed :

$$\begin{aligned} T\frac{\partial y_r}{\partial t} = \frac{\partial y_r}{\partial \tau} &\Rightarrow \frac{\partial \tau}{\partial t} = \tfrac{1}{T} \\ &\Rightarrow t = t_0 + T\tau \end{aligned}$$

If we consider $T \to 0$, the boundary layer system is finally given by :

$$\frac{\partial y_r}{\partial \tau} = g_r(z, y + h(z)) = -y_r \gamma_r C_r(-z_r) \tag{9}$$

Given the definition of the cost function $C(z)$, the origin of (9) is obviously exponentially stable.

5.2 Reduced system (slow dynamics)

The reduced system is obtained by setting $d_r = h_r(z_r)$ in the two first equations of system (8). That is to say that we now study the network model (7) "as if" the E2E controller would converge infinitely fast. The reduced model can be written :

$$\begin{cases} \dot{x} &= G(x)x + Bd' \\ \dot{z}_r &= D_r - d'_r \quad r \in \mathcal{I}_s \end{cases} \quad \text{with} \quad d'_r = \frac{w_r}{C_r(-z_r)} \tag{10}$$

If we write $J = [J_{ij}]$, the Jacobian of this system, it is easy to check that $J_{ii} \leq 0$, $J_{ij} \geq 0 \, \forall i \neq j$. These systems are cooperative. However, it is no longer a compartmental system as (10) is not necessarily positive. It is nonetheless still possible to show global stability using the following results. Indeed, (10) has a first integral with positive gradient $H = \sum_i x_i + \sum_r \dot{z}_r = cst$. It is shown in [8] that cooperative systems with monotone first integral have a unique equilibrium point in H. Moreover, it is shown in [7] that if such a system has an irreducible Jacobian matrix, this unique equilibria is a global attractor. Irreducibility can easily be checked as it is equivalent to strong connectivity of the graph associated to the Jacobian.

5.3 Stability results

Given the exponential stability of systems (9) and (10), standard results from singular perturbation analysis allow to state the following stability result : Their exist $T^* > 0$ such that for all $T < T^*$, the unique equilibria of (8) is exponentially stable. Therefore, it means that for sufficiently small fixed propagation delays or equivalently for sufficiently fast convergence of the controller, the global system (8) is globally stable.

6 Conclusion

A global fluid flow model for the description of packet switched networks has been presented. This model, using the buffer levels as its state vector, is suitable to take into account the buffering delays. It was illustrated that this model which is based on mass conservation laws seems appropriate to study congestion control laws which are themselves rooted in a packet conservation principle. This model was then combined with an AIMD model used in the literature which is viewed as a controller for our network model. Using singular perturbation analysis, the global stability of the closed loop system has been studied.

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