Product Form Solution for Cascade Networks with Intermittent Energy

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***Abstract*—The power needs of digital devices, their installation in locations where it is difficult to connect them to the power grid, and the difficulty of frequently replacing batteries, creates the need to operate digital systems with harvested energy. In such cases, local storage batteries must overcome the intermittent nature of the energy supply, and system performance depends on the intermittent energy supply, possible energy leakage, and system workload. Queueing networks with product form solution are standard tools for analysing interconnected systems. They efficiently predict relevant performance metrics including job queue lengths, throughput, system turnaround times and queueing delays. However, existing queueing network models assume unlimited energy availability, while intermittently harvested energy can affect system performance due insufficient supply of energy. Thus this paper develops a new product form solution for the joint probability distribution of energy availability, and job queue length for an N-node tandem system. Such models can represent production lines in manufacturing systems, supply chains, cascaded repeaters for optical links, or a data link with multiple input data ports that feeds into a switch or server. Our result enables the rigorous computation of all the performance metrics for such systems operating with intermittent energy.**

***Index Terms*—Energy Packet Network, Multi-hop Networks, Simultaneous State Transitions, Energy Harvesting, Product-Form Solution**

I. Introduction

The energy needs of devices in the wired backbone of mobile networks, the Internet, and the Internet of Things (IoT), [1], [2], [3], the need to power them even when they are not plugged into permanent sources of electricity [4], and the inconvenience of changing batteries, has motivated research regarding systems which are powered with harvested energy sources [5], [6].

Because harvested energy will generally be intermittent, the Quality of Service (QoS) of systems that operate with harvested energy will critically depend on the interaction between the amount of energy that is harvested and stored in each device and the workload or traffic that the device processes or forwards [7], [8].

Since the mid-sixties, thanks to Jackson's Theorem [9] and its generalisations [10], efficient analytical techniques are available to analyse the behaviour of multiple interconnected units, if each unit operates with unlimited energy. However a new and major challenge is to model systems with intermittent sources of energy, since energy flow is also random and not always available. When energy is not available at a unit, its processing of jobs or forwarding of data packets (DPs) will be interrupted. Just as DPs and jobs are represented as discrete entities, it is then convenient to model the flow of energy in terms of discrete Energy Packets (EPs), and a battery can be viewed as a "buffer or queue" of EPs.

For such systems, mathematical performance models are needed to combine the effect of both the arrival of energy (EPs) and the flow of DPs when the service units are network nodes, or jobs when the service units are computer servers.

Progress in such analytical models has always happened in steps, and the general results in [9] were preceded by an earlier solution technique that only considered tandem queueing systems [11]. Of course, these earlier models did not consider the issue of intermittent energy sources.

Tandem systems are of great interest in several areas. In industrial engineering they are used to model production lines [12], where each unit represents a workstation in a manufacturing system, or a supply chain [13]. They can also be used to model electronic repeaters in optical transmission lines [14] or customer premises links that service many input ports and feed data to a server or switch.

Thus in this paper, we present the first exact closed form product form analytical solution for the joint probability distribution of job queue length and energy supply for a tandem system composed of any number $N \geq 1$ of units.

In this work, each node or unit is assumed to be powered by a separate source of intermittent energy, and each intermediate unit forwards, to its successor, the jobs or DPs that it has processed. All units also receive jobs or DPs from the outside world, and the $N-th$ unit will send the jobs or DPs it processes out to the outside world. In the sequel, to avoid using different terminologies, a unit will be called a node, and rather than say "DP or job" we shall simply refer to DPs. However our model applies just as well to computer jobs, or to jobs in a manufacturing system, or to items that area being forwarded in a supply chain, and that are processed with the help of intermittent energy in each of the successive nodes.

II. Prior Work

The traditional method for analysing interconnected service systems with N nodes or units, such as computer networks or distributed computer systems, when energy availability is unlimited at the nodes, when they are all connected to a

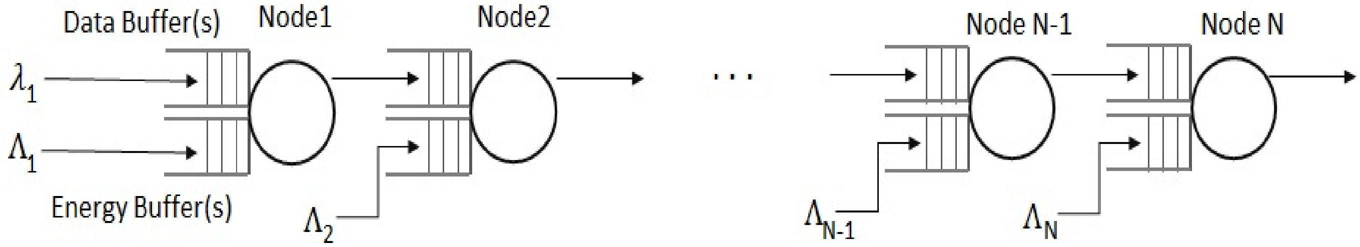

Fig. 1: Cascade network comprised of several store-and-forward layers that forward data packets (DPs) to the devices that collect the data and monitor the sources of the data.

permanent source of electricity, is as a queueing network with Poisson external arrivals of packets to each node i of rate λ_i, and exponential service times with parameter μ_i, in which the movement of data packets is represented by a Markov chain with transition probabilities $P_{ij},\ i,j = 1,\ \dots\ N$, where P_{ij} is the probability that a packet that leaves node i then enters node j. Such a network is also characterised by the probability $d_i = 1 - \sum_j^N P_{ij}$ that a packet departs from the network after being served at node i. In this model, packets entering a node, are queue to wait for service and are served in First-In-First-Out order. This is known as "Jackson's Network" and is widely used as a simple and effective model for multi-hop backbone networks [9], [15]. The product form solution of Jackson's Network states that the joint probability distribution of the N queue lengths of packets at the nodes in steady-state is rigorously expressed as the product (multiplication) of the probability distributions of each individual queue length. Also, the probability distribution of the queue length at node i depends only on the queue's total packet arrival rate Λ^+, and its service rate μ_i. The wide-spread usage of this model, and its generalisations to BCMP networks [10], [16] and G-Networks [17], is due to the rigorous "product form solution" (PFS).

Models where work is conducted in parallel sub-systems and then is synchronised [18], [19], [20], and Petri Nets with synchronisation constraints [21] have also been considered. In [22] a PFS is proved for the flow of work and of control signals in a system of interconnected service units, and it was extended to Petri Nets in [23]. Such techniques have also been used to model Gene Regulatory Networks [24] and Cloud Computing systems [25].

Dynamic control policies for traffic flow in a network with energy harvesting nodes has been considered by several authors. Dynamic control policies for traffic flow in a network with energy harvesting nodes has been considered by several authors. In [26], a mobile base station with a single rechargeable battery is considered and dynamic policies are studied to share power from the same battery, in discrete successive time slots to distinct channels having different traffic rates. Here the power allocation will affect the actual transmitted data rate, and the overall performance metric to be optimised is a function of these effective data rates. In other work [27] that supposes that the topology of a multihop wireless network is fully known and that it does not vary over time, assuming also that the traffic flows are not affected by interference or noise, a control theoretic approach is developed to managing the flow of packets. This work also assumes that the amount of energy and data packets in all nodes is known, that arrival processes of data and energy are time-independent (stationary), and that the information about the backlog of data in a node can be shared with nodes upstream by creating back pressure. In [28], discrete time control models are introduced to maximise the amount of data sensed and forwarded by a sensor network, when the energy it uses is harvested and dynamically allocated to forwarding the data, and assuming that the data forwarding rate depends on the allocated power.

The link between system workload and energy, was recently analysed in [29], where the availability of harvested and stored energy is represented together with the queue length of Data Packets (DPs) in a single node ($N = 1$) system. In this approach energy is discretised in Energy Packets (EP), so that the amount of energy stored in the node's battery is represented as a discrete number of EPs. Thus in the "Energy Packet Network" (EPN) abstraction, a battery is a "buffer queue" for energy, and a network node is represented by two coupled queues, one for DPs and the other for EPs. This approach was generalised to systems with finite DP and EP storage [30], while a two hop ($N = 2$) feed-forward network was studied in [31]. Another approach uses the theory of G-Networks to model the flow of energy as the enabler of service being rendered at a Cloud server [25].

The throughput and power consumption in EPs/sec for a single node with transmission errors due to noise and interference, was derived when multiple EPs are needed to transmit a single DP has been considered in [32]. Other work [30] addresses a single node that consumes energy both for data transmission and for processing jobs.

III. The Cascaded N-Hop Network

The cascaded N-hop model considered in this paper is shown in Figure 1. We assume the data and energy buffers at each node are of unlimited storage capacity, and that one DP (or job) is forwarded using one EP. At node 1, the arrival rate of DPs from outside sources is denoted λ_1, while the remaining nodes are just transit nodes and they do not receive external arrivals of DPs. On the other hand, all nodes i receive EPs at rate Λ_i. We also allow EP leakage at node i, and DP loss due to impatience or errors at rate γ_i.

The leakage rate at node i is μ_i when there are more than one EPs at node i, and is μ_i^0 when there is just one EP at node

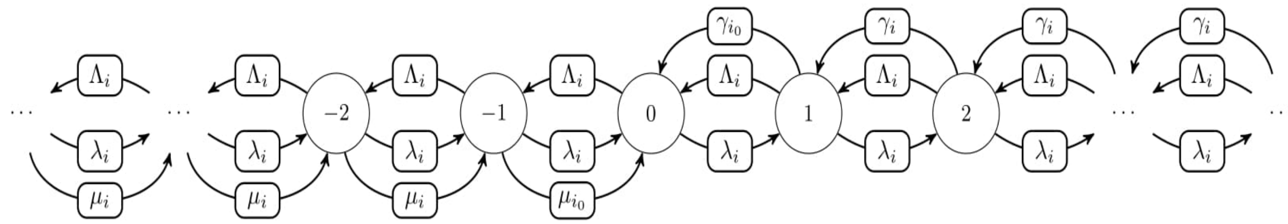

Fig. 2: State transition diagram of Node i.

i. With current electronic technology, the DP transmission time will be in the nanoseconds, while the constitution of a full DP through sensing of external events, the harvesting of a significant amount of energy, the leakage of an EP and the loss of a DP due to impatience or errors will take much longer than a nano-second. Thus, we can assume that the DP forwarding times are negligibly small compared to these other times durations.

The state of node $i \in \{1, \ldots, N\}$ at time t can be represented by the pair (x_i^t, y_i^t) where the first variable represents the backlog of DPs at the node, while the second variable is the amount of energy (in EPs) available at the same node. As with the single node model we must have $x_i^t \, . \, y_i^t = 0$ since if there is both an EP and a DP at a node, the transmission occurs until either all DPs or all EPs are depleted at node i. Thus the state of a node may be represented by a single variable $n_i^t = x_i^t - y_i^t$. If:

- $n_i^t > 0$ then node i has $n_i^t = x_i^t$ DPs waiting to be forwarded, but it does not have the EPs at that node to start the transmission from that node,
- $n_i^t < 0$, then node i has a reserve of $-y_i^t$ EPs, but does not have any DPs to transmit,
- $n_i^t = 0$, then node i does not have any DP and EP in their respective buffers.

The cascaded network is then represented by the vector of positive, negative or zero integers: $\bar{n}^t = (n_1^t, \ldots, n_N^t)$, $t \geq 0$, and $\bar{n}$ denotes a particular value of the vector, so that we study the probability $p(\bar{n}, t) = Prob[\bar{n}^t = \bar{n}]$. Figure (2) shows the state transition diagram of Node i.

Let $\bar{e_i} \triangleq (0, 0, \cdots, 1, \cdots, 0)$ be a vector whose i^{th} element is 1 and other $N-1$ elements are 0. The equilibrium equations for the steady-state probability distribution $\pi(\bar{n})$ for this system are:

$$\pi(\bar{n})\lambda_1 + \pi(\bar{n}) \sum_{i=1}^{N} [\Lambda_i + \gamma_i 1_{n_i>1} + \gamma_i^o 1_{n_i=1} + . \tag{1}$$

$$.\mu_i 1_{n_i<-1} + \mu_i^o 1_{n_i=-1}] =$$

$$\sum_{i=1}^{N} \pi(\bar{n} + e_i)[\ \gamma_i 1_{n_i>0} + \gamma_i^o 1_{n_i=0} + \Lambda_i 1_{n_i<0}\ ] \tag{2}$$

$$+\sum_{i=1}^{N} \pi(\bar{n} - e_i)[\ \mu_i 1_{n_i<0} + \mu_i^o 1_{n_i=0}\ ] + . \tag{3}$$

$$.\pi(\bar{n} - e_1)\lambda_1 1_{n_1>0}$$

$$+\sum_{i=2}^{N} \lambda_1 \pi(\bar{n} - e_1 - \ldots - e_{i-1} - e_i) \prod_{j=1}^{i-1} 1_{n_j \leq 0} 1_{n_i>0} \tag{4}$$

$$+\lambda_1 \pi(\bar{n} - e_1 - \ldots - e_N) \prod_{j=1}^{N} 1_{n_j \leq 0} + . \tag{5}$$

$$.\Lambda_N \pi(\bar{n} + e_N) 1_{n_N \geq 0}$$

$$+\sum_{i=1}^{N-1} \Lambda_i \pi(\bar{n} + e_i - \sum_{k=1}^{N-i} e_{i+k}) 1_{n_i \geq 0} \prod_{k=1}^{N-i} 1_{n_{i+k} \leq 0} \tag{6}$$

$$+\sum_{i=1}^{N-1} \sum_{l=1}^{N-i} \Lambda_i \pi(\bar{n} + e_i - \sum_{k=1}^{l-1} e_{i+k} - e_{i+l}). \tag{7}$$

$$.1_{n_i \geq 0} \prod_{k=1}^{l-1} 1_{n_{i+k} \leq 0} 1_{n_{i+l}>0}.$$

In these equations:

- Line (1) corresponds to the case where there are no arrivals or departures of DPs and EPs, while the first two terms in (2) correspond to the removal of DPs due to impatience, and the third term relates to the arrival of a DP to any Node i where no DPs are being stored.
- The first two terms in (3) correspond to the leakage of EPs, while the third term is due to the arrival of a DP to Node 1 (the only node where DPs can arrive) when Node 1 does not have any EPs.
- The term in (4) corresponds to the arrival of a DP to node 1 when nodes 1 to $i-1$ contain EPs, while Node i does not contain any EPs, so that the DP progresses directly to Node i where it stops to join the DP queue.
- In (5), the first term corresponds to the case where an arriving DP proceeds directly to the exit from Node N because all nodes contain at least one EP. The second term in (5) describes the arrival of an EP to Node N which contains at least one DP which then leaves the network.
- In (6), an EP arrives to Node i containing at least one DP, which then moves all the way to the output of the network because all nodes after node i contain at least one EP.
- Finally (8), it describes the arrival of an EP to Node i when it contains at least one DP; the DP then is able to move through nodes $i+1$ to $i+l-1$ which all contain EPs, but it joins the DP queue at Node $i+l$ which has no EPs.

The equilibrium equations may be rewritten in more compact form as:

$$\pi(\bar{n})\ [\lambda_1 + \sum_{i=1}^{N}(\Lambda_i + \gamma_i 1_{n_i>1} + \gamma_i^0 1_{n_i=1} + . \tag{8}$$

$$.\mu_i 1_{n_i<-1} + \mu_i^0 1_{n_i=-1})]$$

$$= \sum_{i=1}^{N}[\pi(\bar{n}+e_i)(\gamma_i 1_{n_i>0} + \gamma_i^0 1_{n_i=0} + . \tag{9}$$

$$.\Lambda_i 1_{n_i<0} 1_{i\neq N} + \Lambda_N 1_{i=N})]$$

$$+ \sum_{i=1}^{N}[\pi(\bar{n}-e_i)(\mu_i 1_{n_i<0} + \mu_i^0 1_{n_i=0} + . \tag{10}$$

$$.\lambda_1 1_{n_1>0} 1_{i=1})]$$

$$+ \sum_{j=1}^{N-1}[\pi(\bar{n} - \sum_{i=1}^{j+1} e_i)\lambda_1 \prod_{i=1}^{j} 1_{n_i\leq 0}. \tag{11}$$

$$.(1_{n_{1+j=N}} + 1_{n_{j+1}\geq 1} 1_{n_{1+j\neq N}})]$$

$$+ \sum_{j=1}^{N-1}\sum_{i=1}^{j}[\pi(\bar{n}+e_i - \sum_{k=1}^{N-j} e_{i+k})\Lambda_i 1_{n_i\geq 0}(1_{N-j\leq 1}. \tag{12}$$

$$+.1_{N-j\geq 2}\prod_{k=1}^{N-j-1} 1_{n_{i+k}\leq 0})(1_{i=j} + 1_{n_{N+i-j}\geq 1} 1_{i\neq j})\ ]$$

A. The Equilibrium Condition for Energy and Data Flows (EDF)

We can imagine that if the amount of energy that the system harvests is not sufficient to allow the transmission of the incoming flow of DPs, then the backlog of DPs will become infinite. Similarly, if the flow of DPs is not large enough to use the incoming flow of energy, then the backlog of EPs will grow indefinitely. Of course, we also need to include the effect of time-outs for the DPs, and the leakage of the EPs. Let:

$$v_1 = \lambda_1,\ v_{i+1} = \lambda_1 \prod_{l=1}^{i} \frac{\Lambda_i}{\Lambda_i + \gamma_i}, \tag{13}$$

where v_i can be interpreted (see Theorem 2) as the arrival rate of DPs to Node i. The Energy and Data Flow (EDF) condition is then defined as:

$$v_i - \gamma_i = \Lambda_i - \mu_i, \tag{14}$$

where (14) says that the net inflow of DPs, after removal of those that time-out, should be the same as the total inflow of EPs minus the loss of EPs due to leakage.

Theorem 1 Assume that the EDF condition is satisfied. Then the steady-state probability distribution for the system $\pi(\bar{n}) = \lim_{\ t\to\infty} p(\bar{n}, t)$ is given by:

$$\pi(\bar{n}) = \prod_{i=1}^{N}\ \pi_i(n_i),$$

where

$$\pi_i(n_i) = \begin{cases} C_i, & \text{if } n_i = 0,\\ C_i.\frac{v_i}{\Lambda_i+\gamma_i^0}(\frac{v_i}{\Lambda_i+\gamma_i})^{n_i-1}, & \text{if } n_i \geq 1,\\ C_i.\frac{\Lambda_i}{v_i+\mu_i^0}(\frac{\Lambda_i}{v_i+\mu_i})^{-n_i-1}, & \text{if } n_i \leq -1, \end{cases}$$

where $\mu_i^0 = v_i + 2\mu_i$, $\gamma_i^0 = \Lambda_i + 2\gamma_i$, and the normalising constants C_i are:

$$C_i = (1 + \frac{\frac{v_i}{\Lambda_i+\gamma_i^0}}{1 - \frac{v_i}{\Lambda_i+\gamma_i}} + \frac{\frac{\Lambda_i}{v_k+\mu_i^0}}{1 - \frac{\Lambda_i}{v_i+\mu_i}})^{-1}. \tag{15}$$

Using the EDF condition we have:

$$C_i = \frac{2\gamma_i.\mu_i}{2\gamma_i.\mu_i + \Lambda_i.\mu_i + v_i.\gamma_i} \tag{16}$$

$$= \frac{2\gamma_i.\mu_i}{(\gamma_i+\mu_i)(v_i+\mu_i)} = \frac{2\gamma_i.\mu_i}{(\gamma_i+\mu_i)(\Lambda_i+\gamma_i)}. \tag{17}$$

The proof is given in the Appendix.

B. Data Packet Arrival Rates to Nodes

The second result concerns the steady state arrival rate of DPs to each node.

Theorem 2 Denote the steady-state arrival rate of DPs to Node i by α_i, and obviously $\alpha_1 = \lambda_1$. Then $\alpha_i = v_i,\ i = 1,\ \dots, N$.

Proof Let :

$$\nu_i = \sum_{n_i<0}\pi_i(n_i) = C_i\frac{\Lambda_i}{2\gamma_i} = \frac{\Lambda_i\mu_i}{(\gamma_i+\mu_i)(v_i+\mu_i)}, \tag{18}$$

$$\rho_i = \sum_{n_i>0}\pi_i(n_i) = C_i\frac{v_i}{2\mu_i} = \frac{\gamma_i v_i}{(\gamma_i+\mu_i)(\Lambda_i+\gamma_i)}. \tag{19}$$

Then for $i > 1$,

$$\alpha_i = \lambda_1\prod_{j=1}^{i-1}\nu_j + \sum_{j=1}^{i-2}\Lambda_j\rho_j\prod_{k=j+1}^{i-1}\nu_k + \Lambda_{i-1}\rho_{i-1}, \tag{20}$$

$$= v_i\prod_{j=1}^{i-1}\frac{\mu_j}{\gamma_j+\mu_j} + \sum_{j=1}^{i-2}\frac{\gamma_j}{\gamma_j+\mu_j}v_j\frac{\Lambda_j}{\Lambda_j+\gamma_j}. \tag{21}$$

$$.\prod_{k=j+1}^{i-1}\frac{\Lambda_k}{\Lambda_k+\gamma_k}\frac{\mu_k}{\gamma_k+\mu_k} \tag{22}$$

$$+\frac{\gamma_{i-1}}{\gamma_{i-1}+\mu_{i-1}}v_{i-1}\frac{\Lambda_{i-1}}{\Lambda_{i-1}+\gamma_{i-1}}, \tag{23}$$

$$= v_i\prod_{j=1}^{i-1}\frac{\mu_j}{\gamma_j+\mu_j} + v_i\sum_{j=1}^{i-2}\frac{\gamma_j}{\gamma_j+\mu_j}\prod_{k=j+1}^{i-1}\frac{\mu_k}{\gamma_k+\mu_k} \tag{24}$$

$$+v_i\frac{\gamma_{i-1}}{\gamma_{i-1}+\mu_{i-1}}, \tag{25}$$

or denoting $u_i = \frac{\gamma_i}{\gamma_i+\mu_i}$, we have:

$$\frac{\alpha_i}{v_i} = \prod_{j=1}^{i-1}(1-u_j) + u_{i-1} + \sum_{j=1}^{i-2}u_j\prod_{k=j+1}^{i-1}(1-u_k). \tag{26}$$

However, we can easily show by induction on the integer $M \geq 1$ that:

$$\prod_{j=1}^{M}(1-u_j) = 1 - u_M - \sum_{j=1}^{M-1}u_j\prod_{k=j+1}^{M}(1-u_k). \tag{27}$$

Hence the arrival rate of DPs to Node i is $\alpha_i = v_i$, completing the proof.

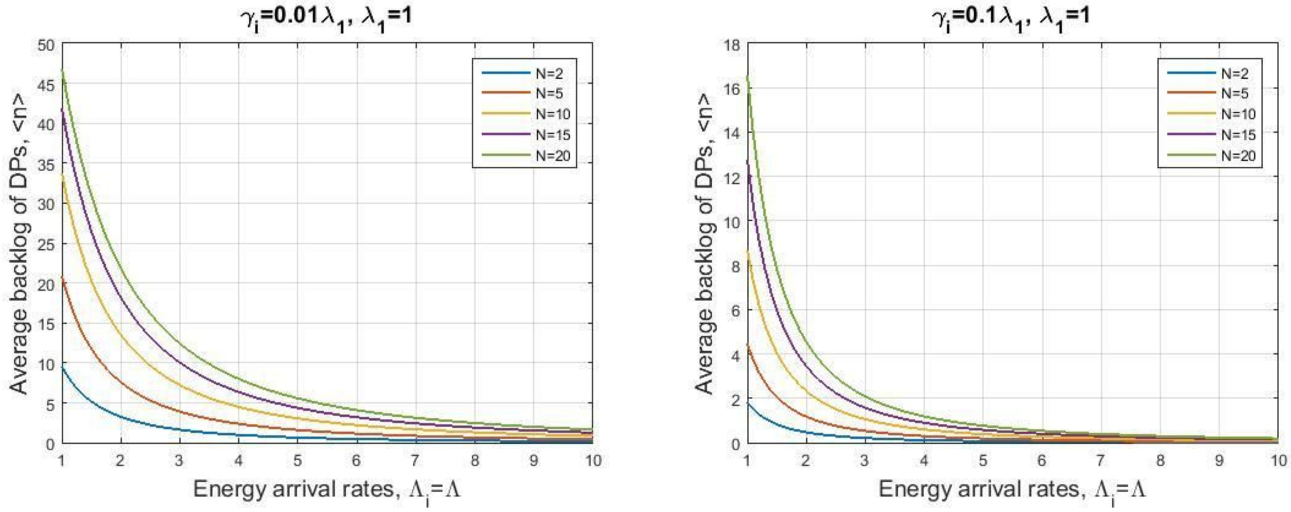

Fig. 3: The total average backlog of DPs at all of the N units, versus the arrival rate of EPs set an identical value at all units $\Lambda_i = \Lambda$. We vary the total number of cascaded units N. The other parameters are $\lambda_1 = 1$, $\gamma_i = 0.01\lambda_1$ (left) and $\gamma_i = 0.1\lambda_1$ (right), respectively. We see that the values of N, Λ and γ_i impact the DP backlog time significantly. Note that the total energy arrival rate to the system is $N\Lambda$.

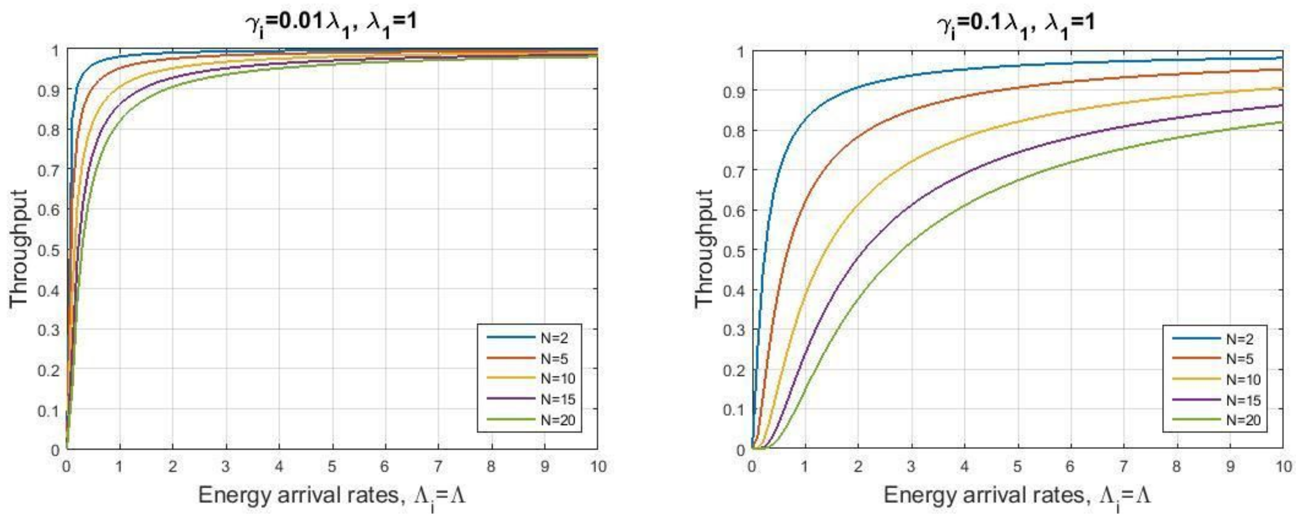

Fig. 4: Throughput versus the EP arrival rate at all units; note that we have set the EP arrival rates to be identical at all nodes with $\Lambda_i = \Lambda$. The number of cascaded nodes or units N is varied. Other parameters are $\lambda_1 = 1$, $\gamma_i = 0.01\lambda_1$ (left) and $\gamma_i = 0.1\lambda_1$ (right), respectively. We see that the values of N, Λ impact the throughput significantly. As the number of nodes increases, the amount of energy per node, needed to "push" the customers or DPs out ofthe network that the throughput reaches 1 is many-fold the DP arrival rate.

IV. Total Backlog of Data Packets

The total average number of DPs in the cascade network is:

$$< n >= \sum_{i=1}^{N} \sum_{n_i>0} n_i \pi_i(n_i) = \sum_{i=1}^{N} \frac{C_i}{2} \frac{R_i}{(1-R_i)^2} \tag{28}$$

$$= \sum_{i=1}^{N} \frac{C_i}{2} \frac{v_i}{\mu_i} [1 + \frac{v_i}{\mu_i}] = \sum_{i=1}^{N} \frac{\gamma_i}{\gamma_i + \mu_i} \frac{v_i}{\mu_i}, \tag{29}$$

where $R_i = \frac{v_i}{v_i+\mu_i}$ and the condition $R_i < 1$ must be satisfied.

Figure (3) shows the average backlog of DPs for different energy arrival rates, and different numbers of nodes N in the cascade network. We assume that external data arrival rate takes unit value $\lambda_1 = 1$, all the nodes have identical EP arrivals $\Lambda_i = \Lambda$ and leakage rates, $\mu_i = 0.1\Lambda_i$, identical DP impatience rates $\gamma_i = c\lambda_1$, with $c = 0.01, \ 0.1$. As one would expect, when the EP arrival rate increases, the average DP backlog decreases significantly since DPs are more rapidly transmitted. Higher DP leakage rates will result in lower overall packet backlogs since more DPs are dropped by the nodes during the transmissions through the network.

Figure (4) shows the throughput for different energy arrival rates, and different numbers of nodes N in the cascade network. We assume same parametric values as in Figure (3). As the number of sensor nodes in the cascaded network increases, the throughput reduces significantly. On the other hand, the increase in EP arrivals elevates the throughput since more data can be served and time-out losses can be reduced.

Also, higher time-out data loss rate causes less throughput as expected.

Parameter	Description
λ_1	External DP arrival rate at Node 1
v_i	Total DP arrival rate at Node i
Λ_i	EP arrival rate at Node i
μ_i	EP loss rate at Node i
γ_i	DP loss rate at Node i
$< n >$	Total average number of DPs
N	Number of nodes in the cascade network

Table I: Parameters used for numerical examples

V. Conclusions

In this paper, we have introduced a mathematical model of a cascaded multi-hop network or a service system where each node gathers energy through harvesting. DPs (or jobs) arrive to the first node and are forwarded hop-by-hop to the output node, as long as there is at least one EP present at each node that is visited. If a DP encounters a node that does not have at least one EP, then the DP must wait for the arrival of enough energy through harvesting at that node. We also assume that EPs are lost at each node due to leakage, and that DPs may also be lost at nodes due to time-outs or errors.

We assume that DPs arrive to the first node according to a Poisson process, and that EPs are harvested at each node according to independent Poission processes. We also make the physically that the time it takes to forward a DP from one node to its immediate neighbour when the node has enough energy, is much shorter than the time it takes to constitute an input DP from sensed data, and the time it takes to harvest an EP that is needed to forward a DP.

Our main result is a previously unknown PFS (product-form solution) for this system. We illustrate the use of this analytical solution by computing the average backlog of DPs and their waiting at each node, and we illustrate this through several numerical results.

Since product form analytical solutions are very useful computational tools in network and computer system performance analysis, and are economical in compute time as compared to discrete event simulations, we expect that the results presented in this paper will be extended in future work to cover more general network topologies. Furthermore, though we have started with Poisson arrivals, we expect that (as with other areas of system performance analysis), these results will lead to further work and that they will be generalised to dependent (rather than independent) inter-arrival times, and to time-varying arrival rates.

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APPENDIX: PROOF OF THEOREM 1

$$\frac{\pi_i(n_i+1)}{\pi_i(n_i)} \triangleq G_i^+ = \begin{cases} \frac{v_i+\mu_i}{\Lambda_i}, & \text{if } n_i < -1 \\ \frac{v_i+\mu_i^0}{\Lambda_i}, & \text{if } n_i = -1 \\ \frac{v_i}{\Lambda_i+\gamma_i^0}, & \text{if } n_i = 0 \\ \frac{v_i}{\Lambda_i+\gamma_i}, & \text{if } n_i > 0 \end{cases}$$

$$\frac{\pi_i(n_i-1)}{\pi_i(n_i)} \triangleq G_i^- = \begin{cases} \frac{\Lambda_i+\gamma_i}{v_i}, & \text{if } n_i > 1 \\ \frac{\Lambda_i+\gamma_i^0}{v_i}, & \text{if } n_i = 1 \\ \frac{\Lambda_i}{v_i+\mu_i^0}, & \text{if } n_i = 0 \\ \frac{\Lambda_i}{v_i+\mu_i}, & \text{if } n_i < 0 \end{cases}$$

$$\prod_{i=1}^{M} G_i^- 1_{n_i \le 0} = \frac{v_{M+1}}{v_1}\prod_{i=1}^{M} f(n_i),$$

$$G_i^+\Lambda_i 1_{n_i\ge 0} = v_{i+1}h(n_i),$$

where

$$f(n_i) = \frac{1_{n_i=0}}{2} + 1_{n_i<0}$$

and

$$h(n_i) = \frac{1_{n_i=0}}{2} + 1_{n_i>0}.$$

Dividing both sides of the equilibrium equation by $\pi(\bar{n})$ we have:

$$[\lambda_1 + \sum_{i=1}^{N}(\Lambda_i + \gamma_i 1_{n_i>1} + \gamma_i^0 1_{n_i=1} + . \quad (30)$$

$$.\mu_i 1_{n_i<-1} + \mu_i^0 1_{n_i=-1})]$$

$$= \sum_{i=1}^{N}[G_i^+(\gamma_i 1_{n_i>0} + \gamma_i^0 1_{n_i=0} + . \quad (31)$$

$$.\Lambda_i 1_{n_i<0}1_{i\neq N} + \Lambda_N 1_{i=N})]$$

$$+\sum_{i=1}^{N}[G_i^-(\mu_i 1_{n_i<0} + \mu_i^0 1_{n_i=0} + \lambda_1 1_{n_1>0}1_{i=1})] \quad (32)$$

$$+\sum_{j=1}^{N-1}[\prod_{i=1}^{j+1}G_i^-\lambda_1\prod_{i=1}^{j}1_{n_i\le 0}(1_{1+j=N} + 1_{n_{j+1}\ge 1}1_{1+j\neq N})] \quad (33)$$

$$+\sum_{j=1}^{N-1}\sum_{i=1}^{j}[G_i^+\prod_{k=1}^{N-j}G_{k+i}^-(\Lambda_i 1_{n_i\ge 0}(1_{N-j\le 1} + 1_{N-j\ge 2}. \quad (34)$$

$$.\prod_{k=1}^{N-j-1}1_{n_{i+k}\le 0})(1_{i=j} + 1_{n_{N+i-j}\ge 1}1_{i\neq j}))]. \quad (35)$$

Rewriting (31) we have:

$$= G_N^+\Lambda_N 1_{n_N\ge 0} + \sum_{i=1}^{N}[\frac{v_i\gamma_i}{\Lambda_i+\gamma_i}1_{n_i>0} + . \quad (36)$$

$$.\frac{v_i\gamma_i^0}{\Lambda_i+\gamma_i^0}1_{n_i=0} + (v_i+u_i^0)1_{n_i=-1} + (v_i+u_i)1_{n_i<-1}],$$

and rewriting (32) as:

$$G_1^-\lambda_1 1_{n_1>0} + \sum_{i=1}^{N}[\frac{\Lambda_i\mu_i}{v_i+\mu_i}1_{n_i<0} + \frac{\Lambda_i\mu_i^0}{v_i+\mu_i^0}1_{n_i=0}], \quad (37)$$

we sum (36) and (37):

$$= G_1^-\lambda_1 1_{n_1>0} + G_N^+\Lambda_N 1_{n_N\ge 0} + . \quad (38)$$

$$.\sum_{i=1}^{N}[\frac{v_i\gamma_i}{\Lambda_i+\gamma_i}1_{n_i>0} + (\frac{v_i\gamma_i^0}{\Lambda_i+\gamma_i^0} + \frac{\Lambda_i\mu_i^0}{v_i+\mu_i^0})1_{n_i=0} + .$$

$$.(v_i + u_i^0 + \frac{\Lambda_i\mu_i}{v_i+\mu_i})1_{n_i=-1} + .$$

$$.(v_i + u_i + \frac{\Lambda_i\mu_i}{v_i+\mu_i})1_{n_i<-1}],$$

where

$$\frac{v_i\gamma_i^0}{\Lambda_i+\gamma_i^0} + \frac{\Lambda_i\mu_i^0}{v_i+\mu_i^0} = \frac{v_i(\Lambda_i+2\gamma_i) + \Lambda_i(v_i+2\mu_i)}{2(\Lambda_i+\gamma_i)}$$

$$= v_i + \frac{\Lambda_i\mu_i}{\Lambda_i+\gamma_i}, \quad (39)$$

and

$$\frac{v_i\gamma_i}{\Lambda_i+\gamma_i} = v_i + \frac{\Lambda_i\mu_i}{\Lambda_i+\gamma_i} + \frac{v_i\gamma_i}{\Lambda_i+\gamma_i} - v_i - \frac{\Lambda_i\mu_i}{\Lambda_i+\gamma_i}$$

$$= v_i + \frac{\Lambda_i\mu_i}{\Lambda_i+\gamma_i} + \frac{v_i\gamma_i - v_i\Lambda_i - v_i\gamma_i - \Lambda_i\mu_i}{\Lambda_i+\gamma_i}$$

$$= v_i - \Lambda_i + \frac{\Lambda_i\mu_i}{\Lambda_i+\gamma_i}. \quad (40)$$

Now by inserting (39) and (40) into (38) we have:

$$= G_1^- \lambda_1 1_{n_1>0} + G_N^+ \Lambda_N 1_{n_N \geq 0} + \sum_{i=1}^{N} [\Lambda_i + . \quad (41)$$
$$.\frac{v_i \gamma_i}{\Lambda_i + \gamma_i} + \mu_i^0 1_{n_i=-1} + \mu_i 1_{n_i<-1} - \Lambda_i 1_{n_i>0}],$$

so that the equilibrium equation reduces to:

$$\lambda_1 + \sum_{i=1}^{N} [\gamma_i 1_{n_i>1} + \gamma_i^0 1_{n_i=1}] \quad (42)$$
$$= G_1^- \lambda_1 1_{n_1>0} + G_N^+ \Lambda_N 1_{n_N \geq 0} + . \quad (43)$$
$$\sum_{i=1}^{N} [\frac{v_i \gamma_i}{\Lambda_i + \gamma_i} - \Lambda_i 1_{n_i>0}]$$
$$+ \sum_{j=1}^{N-1} [\prod_{i=1}^{j+1} G_i^- \lambda_1 \prod_{i=1}^{j} 1_{n_i \leq 0}. \quad (44)$$
$$(1_{1+j=N} + 1_{n_{j+1} \geq 1} 1_{1+j \neq N})]$$
$$+ \sum_{j=1}^{N-1} \sum_{i=1}^{j} [G_i^+ \prod_{k=1}^{N-j} G_{k+i}^- \Lambda_i 1_{n_i \geq 0} (1_{N-j \leq 1} + 1_{N-j \geq 2}. \quad (45)$$
$$. \prod_{k=1}^{N-j-1} 1_{n_{i+k} \leq 0})(1_{i=j} + 1_{n_{N+i-j} \geq 1} 1_{i \neq j})].$$

Now consider (44):

$$= \lambda_1 G_N^- \prod_{i=1}^{N-1} G_i^- 1_{n_i \leq 0} \quad (46)$$
$$+ \sum_{j=1}^{N-2} [\lambda_1 G_{j+1}^- 1_{n_{j+1} \geq 1} \prod_{i=1}^{j} G_i^- 1_{n_i \leq 0}] \quad (47)$$
$$= v_N G_N^- \prod_{i=1}^{N-1} f(n_i) \quad (48)$$
$$+ \sum_{j=1}^{N-2} [G_{j+1}^- v_{j+1} 1_{n_{j+1} \geq 1} \prod_{i=1}^{j} f(n_i)]. \quad (49)$$

Also consider (45):

$$= G_{N-1}^+ \ G_N^- \ \Lambda_{N-1} 1_{N-1 \geq 0} \quad (50)$$
$$+ \sum_{i=1}^{N-2} [G_i^+ \ G_{i+1}^- \ \Lambda_i 1_{i \geq 0} 1_{i+1 \geq 1}] \quad (51)$$
$$+ \sum_{j=2}^{N-2} \sum_{i=1}^{j-1} [G_i^+ \prod_{k=1}^{N-j} G_{k+i}^- \Lambda_i 1_{n_i \geq 0}. \quad (52)$$
$$. \prod_{k=1}^{N-j-1} l_{n_{i+k} \leq 0} \ 1_{n_{N+i-j} \geq 1}]$$
$$+ \sum_{j=1}^{N-2} [G_j^+ \prod_{k=1}^{N-j} G_{k+j}^- \Lambda_j 1_{n_j \geq 0} \prod_{k=1}^{N-j-1} l_{n_{j+k} \leq 0}] \quad (53)$$
$$= v_N \ G_N^- \ h(n_{N-1}) \quad (54)$$
$$+ \sum_{i=1}^{N-2} v_{i+1} \ G_{i+1}^- \ 1_{n_{i+1} \geq 1} \ h(n_i) \quad (55)$$
$$+ \sum_{j=2}^{N-2} \sum_{i=1}^{j-1} [G_i^+ \Lambda_i 1_{n_i \geq 0} \ G_{N-j+i}^- 1_{n_N - j+i \geq 1}. \quad (56)$$
$$. \prod_{k=1}^{N-j-1} G_{k+i}^- 1_{n_k + i \leq 0}]$$
$$+ \sum_{j=1}^{N-2} [G_N^- G_j^+ \Lambda_j 1_{n_j \geq 0} \prod_{k=1}^{N-j-1} G_{k+j}^- l_{n_{j+k} \leq 0}] \quad (57)$$
$$= v_N \ G_N^- \ h(n_{N-1})) \quad (58)$$
$$+ \sum_{i=1}^{N-2} [v_{i+1} \ G_{i+1}^- \ 1_{n_{i+1} \geq 1} \ h(n_i)] \quad (59)$$
$$+ \sum_{j=2}^{N-2} \sum_{i=1}^{j-1} [v_{i+1} h(n_i) \ G_{N-j+i}^- 1_{n_{N-j+i} \geq 1}. \quad (60)$$
$$. \frac{v_{N-j+i}}{v_{i+1}} \prod_{k=1}^{N-j-1} f(n_{k+i})]$$
$$+ \sum_{j=1}^{N-2} [G_N^- \ v_{j+1} h(n_j) \frac{v_N}{v_{j+1}} \prod_{k=1}^{N-j-1} f(n_{k+j})] \quad (61)$$

$$= v_N \ G_N^- \ h(n_{N-1}) \quad (62)$$
$$+ \sum_{i=1}^{N-2} [v_{i+1} \ G_{i+1}^- \ 1_{n_{i+1} \geq 1} \ h(n_i)] \quad (63)$$
$$+ \sum_{j=2}^{N-2} \sum_{i=1}^{j-1} [v_{N-j+i} G_{N-j+i}^- 1_{n_{N-j+i} \geq 1}. \quad (64)$$
$$.h(n_i) \prod_{k=1}^{N-j-1} f(n_{k+i})]$$
$$+ v_N G_N^- \sum_{j=1}^{N-2} [h(n_j) \prod_{k=1}^{N-j-1} f(n_{k+j})]. \quad (65)$$

The summation of (62) and (65):

$$v_N\ G_N^- h(n_{N-1}) + v_N G_N^- \sum_{j=1}^{N-2} [h(n_j) \prod_{k=1}^{N-j-1} f(n_{k+j})] \quad (66)$$

$$= v_N G_N^- \sum_{j=1}^{N-1} [h(n_j) \prod_{k=1}^{N-j-1} f(n_{k+j})] \quad (67)$$

$$= v_N G_N^- \sum_{j=1}^{N-1} [(1-f(n_j)) \prod_{k=1}^{N-j-1} f(n_{k+j})] \quad (68)$$

$$= v_N G_N^- \sum_{j=1}^{N-1} [\prod_{k=1}^{N-j-1} f(n_{k+j}) - f(n_j) \prod_{k=1}^{N-j-1} f(n_{k+j})] \quad (69)$$

$$= v_N G_N^- \sum_{j=1}^{N-1} [\prod_{k=j+1}^{N-1} f(n_k) - f(n_j) \prod_{k=j+1}^{N-1} f(n_k)] \quad (70)$$

$$= v_N G_N^- (1 - \prod_{k=1}^{N-1} f(n_k)). \quad (71)$$

The summation of (48) and (71):

$$v_N G_N^- \prod_{i=1}^{N-1} f(n_i) + v_N G_N^- (1 - \prod_{k=1}^{N-1} f(n_k)) = v_N G_N^-. \quad (72)$$

After these simplifications, we may re-write the equilibrium equation as:

$$\lambda_1 + \sum_{i=1}^{N} (\gamma_i 1_{n_i>1} + \gamma_i^0 1_{n_i=1}) \quad (73)$$

$$= G_1^- \lambda_1 1_{n_1>0} + G_N^+ \Lambda_N 1_{n_N \geq 0} + . \quad (74)$$

$$. \sum_{i=1}^{N} [\frac{v_i \gamma_i}{\Lambda_i + \gamma_i} - \Lambda_i 1_{n_i>0}]$$

$$+ \sum_{j=1}^{N-2} [G_{j+1}^- v_{j+1} 1_{n_{j+1} \geq 1} \prod_{i=1}^{j} f(n_i)] \quad (75)$$

$$+ v_N G_N^- \quad (76)$$

$$+ \sum_{i=1}^{N-2} [v_{i+1}\ G_{i+1}^-\ 1_{n_{i+1} \geq 1}\ h(n_i)] \quad (77)$$

$$+ \sum_{j=2}^{N-2} \sum_{i=1}^{j-1} [v_{N-j+i} G_{N-j+i}^- 1_{n_{N-j+i} \geq 1}. \quad (78)$$

$$.h(n_i) \prod_{k=1}^{N-j-1} f(n_{k+i})].$$

The summation of (74) and (76):

$$= G_1^- \lambda_1 1_{n_1>0} + G_N^+ \Lambda_N 1_{n_N \geq 0} + \sum_{i=1}^{N} \frac{v_i \gamma_i}{\Lambda_i + \gamma_i} - . \quad (79)$$

$$. \sum_{i=1}^{N} \Lambda_i 1_{n_i>0} + v_N G_N^- 1_{n_N>0} + v_N G_N^- 1_{n_N \leq 0},$$

where

$$G_N^+ \Lambda_N 1_{n_N \geq 0} + v_N G_N^- 1_{n_N \leq 0} = \frac{v_N \Lambda_N}{\Lambda_N + \gamma_N}, \quad (80)$$

and

$$\frac{v_N \Lambda_N}{\Lambda_N + \gamma_N} + \sum_{i=1}^{N} \frac{v_i \gamma_i}{\Lambda_i + \gamma_i} = \lambda_1. \quad (81)$$

Thus the equilibrium equation has been simplified to:

$$\sum_{i=1}^{N} (\gamma_i 1_{n_i>1} + \gamma_i^0 1_{n_i=1} + \Lambda_i 1_{n_i>0}) - . \quad (82)$$

$$.G_1^- \lambda_1 1_{n_1>0} - G_N^- v_N 1_{n_N>0}$$

$$= \sum_{j=1}^{N-2} G_{j+1}^- v_{j+1} 1_{n_{j+1} \geq 1} \prod_{i=1}^{j} f(n_i) \quad (83)$$

$$+ \sum_{i=1}^{N-2} v_{i+1}\ G_{i+1}^-\ 1_{n_{i+1} \geq 1}\ h(n_i) \quad (84)$$

$$+ \sum_{j=2}^{N-2} \sum_{i=1}^{j-1} v_{N-j+i} G_{N-j+i}^- 1_{n_{N-j+i} \geq 1}. \quad (85)$$

$$.h(n_i) \prod_{k=1}^{N-j-1} f(n_{k+i})$$

Define $\Theta_i \triangleq G_i^-\ v_i\ 1_{n_i \geq 1}$ so that (85) becomes:

$$= \Theta_{N-1} h(n_1) f(n_2) f(n_3) \cdots f(n_{N-2}) \quad (86)$$

$$+ \Theta_{N-1} h(n_2) f(n_3) f(n_4) \cdots f(n_{N-2}) \quad (87)$$

$$\vdots$$

$$+ \Theta_{N-1} h(n_N - 4) f(n_{N-3}) f(n_{N-2}) \quad (88)$$

$$+ \Theta_{N-1} h(n_N - 3) f(n_{N-2}) \quad (89)$$

$$+ \Theta_{N-2} h(n_1) f(n_2) f(n_3) \cdots f(n_{N-3}) \quad (90)$$

$$+ \Theta_{N-2} h(n_2) f(n_3) f(n_4) \cdots f(n_{N-3}) \quad (91)$$

$$\vdots$$

$$+ \Theta_{N-2} h(n_N - 5) f(n_{N-4}) f(n_{N-3}) \quad (92)$$

$$+ \Theta_{N-2} h(n_N - 4) f(n_{N-3}) \quad (93)$$

$$\vdots$$

$$+ \Theta_4 h(n_1) f(n_2) f(n_3) \quad (94)$$

$$+ \Theta_4 h(n_2) f(n_3) \quad (95)$$

$$+ \Theta_3 h(n_1) f(n_2). \quad (96)$$

Since $h(n_i) = 1 - f(n_i)$ we may re-write from (86) to (89) as:

$$\Theta_{N-1} f(n_2) f(n_3) \cdots f(n_{N-2}) - . \quad (97)$$

$$.\Theta_{N-1} f(n_1) f(n_2) f(n_3) \cdots f(n_{N-2})$$

$$+ \Theta_{N-1} f(n_3) f(n_4) \cdots f(n_{N-2}) - . \quad (98)$$

$$.\Theta_{N-1} f(n_2) f(n_3) f(n_4) \cdots f(n_{N-2})$$

$$\vdots$$

$$+ \Theta_{N-1} f(n_{N-3}) f(n_{N-2}) - . \quad (99)$$

$$.\Theta_{N-1} f(n_N - 4) f(n_{N-3}) f(n_{N-2})$$

$$+ \Theta_{N-1} f(n_{N-2}) - \Theta_{N-1} f(n_N - 3) f(n_{N-2}) \quad (100)$$

$$= \Theta_{N-1} f(n_{N-2}) - \Theta_{N-1} \prod_{k=1}^{N-2} f(n_i). \quad (101)$$

Similarly, we consider the Θ_i's for $i \in \{3, \cdots, N-1\}$ and have:

$$\sum_{j=2}^{N-2}\sum_{i=1}^{j-1}[v_{N-j+i}G^{-}_{N-j+i}1_{n_{N-j+i}\geq 1}. \tag{102}$$

$$.h(n_i)\prod_{k=1}^{N-j-1}f(n_{k+i})]$$

$$= \sum_{j=3}^{N-1}[\Theta_j f(n_{j-1})] - \sum_{j=3}^{N-1}[\Theta_j \prod_{k=1}^{j-1} f(n_k)]. \tag{103}$$

Thus, the equilibrium equation has now been reduced to:

$$\sum_{i=1}^{N}[G^{-}_i v_i 1_{n_i\geq 1}] - G^{-}_1\lambda_1 1_{n_1>0} - G^{-}_N v_N 1_{n_N>0} \tag{104}$$

$$= \sum_{j=1}^{N-2}[G^{-}_{j+1}v_{j+1}1_{n_{j+1}\geq 1}\prod_{i=1}^{j}f(n_i)] \tag{105}$$

$$+ \sum_{i=1}^{N-2}[v_{i+1}\ G^{-}_{i+1}\ 1_{n_{i+1}\geq 1}\ h(n_i)] \tag{106}$$

$$+ \sum_{j=3}^{N-1}[\Theta_j f(n_{j-1})] - \sum_{j=3}^{N-1}[\Theta_j \prod_{k=1}^{j-1} f(n_k)], \tag{107}$$

or

$$\sum_{i=2}^{N-1}\Theta_i \tag{108}$$

$$= \sum_{j=3}^{N-1}[\Theta_j \prod_{i=1}^{j-1} f(n_i)] + \Theta_2 f(n_1) \tag{109}$$

$$+ \sum_{i=3}^{N-1}[\Theta_i h(n_{i-1})] + \Theta_2 h(n_1) \tag{110}$$

$$+ \sum_{j=3}^{N-1}[\Theta_j f(n_{j-1})] - \sum_{j=3}^{N-1}[\Theta_j \prod_{k=1}^{j-1} f(n_k)], \tag{111}$$

where

$$\sum_{i=3}^{N-1}[\Theta_i h(n_{i-1})] + \sum_{j=3}^{N-1}[\Theta_j f(n_{j-1})] = \sum_{i=3}^{N-1}\Theta_i, \tag{112}$$

and

$$\Theta_2 f(n_1) + \Theta_2 h(n_1) = \Theta_2. \tag{113}$$

So finally we see that the left and right hand sides of the equilibrium equation cancel each other, completing the proof.