Search in Non-Homogenous Random Environments*

[Extended Abstract]

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Summary Using a mixed jump and diffusion model, we compute the time and energy needed to find an object placed at a finite distance D from a searcher's initial location within an infinite non-homogenous search space, assuming that the searcher has imprecise information about where and how to search, and also that the searcher may be blocked or destroyed during the search. This problem arises in large wired or wireless networks with imprecise routing tables and packet losses [4, 8, 10, 13], in large databases with uncertain or approximately represented data such as the content of images [5, 14], and in the search by robots in hostile environments such as minefields [7].

Introduction An animal's search for prey was modelled in [9, 12] when the predator renews its energy reserve during the search. Randomly connected finite graphs in [11] represent search in a computer network or a system of roads. In [10] it was shown that the time it takes a data packet to travel from a source to a destination node in an infinitely large and unreliable network is finite on average, if a time-out mechanism destroys the packet after a predetermined time, replacing it with a new one that starts at the source proceeding at random and independently of its predecessor. This was generalised [15] to N searchers which are simultaneously, but independently sent out in the quest for the same object. Most of the literature considers homogenous search spaces, and in this paper we develop a mixed analytical-numerical method for an infinite random *non-homogenous medium* that generalises the work in [15] obtaining expressions for the average time and energy that it takes the searcher to eventually find the object it is seeking. An interesting phase transition is exhibited concerning the eventual success of the search depending on the relative speed of approach of the searcher and the intensity of events which block the searcher's progress.

The Model Although traditionally most models in computer systems and networks are discrete [3], here we consider a continuous distance $Y(t)$ of the searcher to the object at time $t \geq 0$. The searcher starts at $Y(0) = D$ and the search ends at time $T = \inf\{t : Y(t) = 0\}$. If the random variable $s(t)$ represents the state of the searcher, $s(t) \in \{\mathbf{S}, \mathbf{W}, \mathbf{P}, \ ...\}$, then $s(t) \in \mathbf{S}$ if the search is proceeding with the search and its distance from the destination is $Y(t) > 0$. The probability density function of $Y(t)$ is denoted $f(z,t)dz = P[z < Y(t) \leq z + dz, \ s(t) = \mathbf{S}]$. $s(t) \in \mathbf{W}$ if the searcher's life-span has ended, and so has its search. This can happen because the searcher was destroyed or became lost, and the source was informed via the time-out. After an additional exponentially distributed delay of parameter μ, meant to avoid mistakes in assuming that the searcher is "dead", a new searcher is placed at the source and a new search immediately begins. We write $W(t) = P[s(t) = \mathbf{W}]$. $s(t) \in \mathbf{L}$ if the searcher is destroyed or lost, and the search is interrupted until a new searcher can be sent out. The time spent in this state is exponentially distributed with parameter r which is the same parameter as that of the "time-out" or life-span, since the source realises that the searcher is lost or destroyed via the life-span or time-out effect. At the end of this exponentially distributed time, the searcher is handled just as if it has "died", and we denote $L(t) = P[s(t) = \mathbf{L}]$. $s(t) \in \mathbf{P}$ if the searcher has reached its destination, i.e. it has found the object it sought and the search process ends. However, as an artefact to construct an indefinitely repeating recurrent process, after one time unit the search process restarts at the source and a new searcher is sent out. We will use the notation $P(t) = P[s(t) = \mathbf{P}]$. Notice that the process repeats itself indefinitely. If $E[T]$ is the average time that it takes from any successive start of the search until the first instance when state $\mathbf{P}$ is reached again, and $P(t)$ is the probability that the model we have just described is in state $\mathbf{P}$ at time $t \geq 0$, and $P = \lim_{t\to\infty} P(t)$, then $P = \frac{1}{1+E[T]}, \ E[T] = P^{-1} - 1$. During the searcher's travel in state $\mathbf{S}$ while $\{Y(t) = z > 0\}$ the following events can occur in the time interval $[t, t+\Delta t[$. With probability $\lambda(z)\Delta t + o(\Delta t)$ the searcher is destroyed or lost, and enters state $\mathbf{L}$. From that state it enters state $\mathbf{W}$ after an exponentially distributed delay of parameter r. With probability $r\Delta t + o(\Delta t)$ the searcher's life-span runs out and it enters state $\mathbf{W}$. Note that $1/r$ is the average life-span. As indicated earlier, when it enters state $\mathbf{W}$, after an additional delay of average value $1/\mu$, the searcher is replaced with a new one at the source. The average rate per unit time at which the searcher approaches the object being sought when it is at distance z is $b(z)$, and the variance of the distance travelled in the interval $[t, t+\Delta t[$ is denoted by

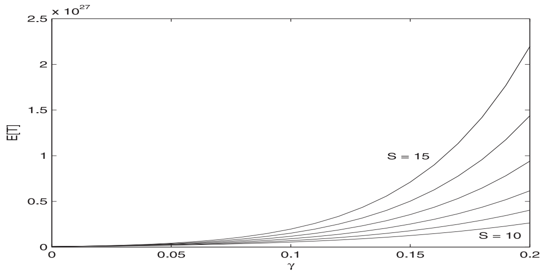

Figure 1: Average search time $E[T]$ versus $\gamma = \lambda_1 - \lambda_2$ for $S = 10 - 15$ with a step size of 1.

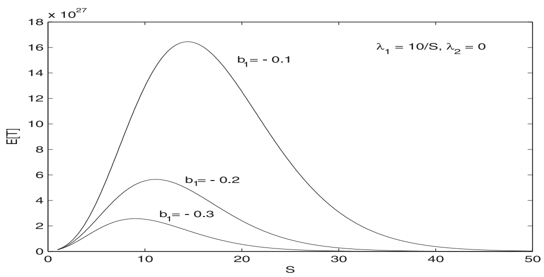

Figure 2: Average search time $E[T]$ versus S when $\lambda_1 = 10/S$ for different values of b_1. The optimum protection area needed becomes smaller so that λ_1 increases when the search speed increases.

$c(z)\Delta t$ so that:

$$b(z) = \lim_{\Delta t \to 0} \frac{E[Y_{t+\Delta t} - Y_t | Y_t = z]}{\Delta t},$$

$$c(z) = \lim_{\Delta t \to 0} \frac{E[(Y_{t+\Delta t} - Y_t)^2 - (E[Y_{t+\Delta t} - Y_t])^2 | Y_t = z]}{\Delta t}$$

When $b(z) < 0$, on average the searcher gets closer over time to the object being sought. The searcher's location probability density function $f(z,t)$ at time $t \geq 0$, satisfies the diffusion equation [1] as in models used previously for packet flow in communication traffic flow in transportation systems or packet flows in communication systems [2, 6]. The equations that $f(z,t)dz$, $z > 0$, and the probability masses $L(t)$, $W(t)$ and $P(t)$, $t \geq 0$ satisfy are:

$$\begin{aligned}
\frac{\partial f(z,t)}{\partial t} =& \frac{1}{2}\frac{\partial^2[c(z)f(z,t)]}{\partial z^2} - \frac{\partial[b(z)f(z,t)]}{\partial z} \\
& - (\lambda(z)+r)f(z,t) + [P(t)+\mu W(t)]\delta(z-D) \\
\frac{dL(t)}{dt} =& - rL(t) + \int_0^\infty \lambda(z)f(z,t)dz \\
\frac{dW(t)}{dt} =& - \mu W(t) + r[L(t) + \int_0^\infty f(z,t)dz] \\
\frac{dP(t)}{dt} =& - P(t) + \lim_{z\to 0^+} \big[\frac{1}{2}\frac{\partial[c(z)f(z,t)]}{\partial z} - b(z)f(z,t)\big] \\
1 =& P(t) + W(t) + L(t) + \int_0^\infty f(z,t)dz
\end{aligned}$$

where the local behaviour of the searcher is captured in the drift $b(z)$, instantaneous variance $c(z)$ as well as loss parameter $\lambda(z)$. This is equivalent to also letting the time-out parameter r be location dependent because it could be included in $\lambda(z)$. We simplify the model by considering a finite unbounded number of "segments" having different parameters for the Brownian motion describing the searcher's movement as a function of distance to the object, while within each segment the parameters are the same. The first segment is in immediate proximity of the object being sought, starting at distance $z = 0$. Each segment may have different size, and there are a total of $m < \infty$ segments. By choosing as many segments as we wish, and letting each segment be as small as needed (all segments need not be of the same length), we can approximate closely any physical situation that arises where the searcher's motion characteristics vary over the distance of the searcher to the object being sought. This representation leads to a neat algebraic "product form" for the average search time, providing a useful analytic form that offers an intuitive representation of the analytical results. Let $0 \leq Z_k < \infty$ be the boundary between the k-th and $(k+1)$-th segments with $Z_0 = 0$. The last segment goes from Z_{m-1} to $+\infty$, and we assume that both m and Z_{m-1} are finite but unbounded. Thus for greater accuracy in representing the search we can take as many segments as we wish, and they may be as small as needed, but they are all finite except the last segment. Thus for $1 \leq k \leq m$, the k-th segment represents the range of distances $Z_{k-1} \leq z < Z_k$, and let $S_k = Z_k - Z_{k-1}$ denote its size. We use n to denote the segment number in which the source point of the search is located, i.e. $Z_{n-1} < D \leq Z_n$. The stationary solution of the location dependent diffusion equation for any segment $k \neq n$ is then:

$$0 = \frac{c_k}{2}\frac{d^2 f_k(z)}{dz^2} - b_k \frac{df_k(z)}{dz} - (\lambda_k + r)f_k(z) \qquad (1)$$

while the equation for the segment where the source is located is:

$$-[P+\mu W]\delta(z-D) = \frac{c_n}{2}\frac{d^2 f_n(z)}{dz^2} - b_n \frac{df_n(z)}{dz} - (\lambda_n + r)f_n(z) \qquad (2)$$

Main Result Let u_k, v_k be the positive and negative real roots of the characteristic polynomial of the stationary diffusion equation for the k-th segment:

$$u_k, v_k = \frac{b_k \pm \sqrt{b_k^2 + 2c_k(\lambda_k + r)}}{c_k}$$

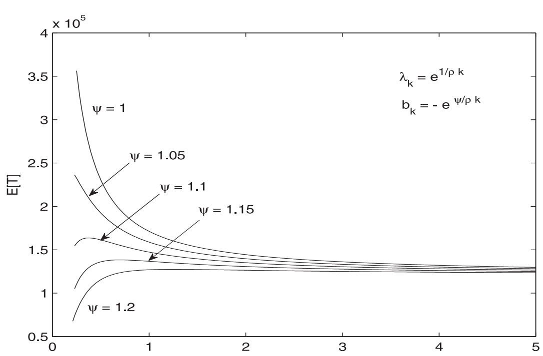

Figure 3: Average search time $E[T]$ versus ρ when $\lambda_k = e^{1/(\rho k)}$ and $b_k = -e^{\psi/(\rho k)}$ for different values of ψ; $c_k = 1$, $D = 10$, $r = 0.05$, $\mu = 0.025$ and $S_k = 1,\ k < m = 20$.

Then the total average search time obtained by solving for P so that $E[T] = P^{-1} - 1$:

$$E[T] = \left(\frac{1}{r} + \frac{1}{\mu}\right) \times$$
$$[\sqrt{\frac{b_n^2 + 2c_n(\lambda_n + r)}{b_1^2 + 2c_1(\lambda_1 + r)}} \frac{\overline{A}_n \overline{G}_n e^{u_n S_n} - \overline{B}_n \overline{F}_n e^{v_n S_n}}{\overline{G}_n e^{u_n(Z_n - D)} + \overline{F}_n e^{v_n(Z_n - D)}} - 1] \quad (3)$$

where the remaining parameters are as follows. Let:

$$\alpha_k^- = \frac{c_k u_k - c_{k-1} v_{k-1}}{c_k(u_k - v_k)}, \quad \beta_k^- = \frac{c_k u_k - c_{k-1} u_{k-1}}{c_k(u_k - v_k)}$$
$$\alpha_k^+ = \frac{c_k u_k - c_{k+1} v_{k+1}}{c_k(u_k - v_k)}, \quad \beta_k^+ = \frac{c_k u_k - c_{k+1} u_{k+1}}{c_k(u_k - v_k)} \quad (4)$$

Then set $\overline{A}_1 = 1$ and $\overline{B}_1 = -1$ and for $2 \le k \le n$ compute:

$$\begin{bmatrix} \overline{A}_k \\ \overline{B}_k \end{bmatrix} = \begin{bmatrix} \alpha_k^- & \beta_k^- \\ 1 - \alpha_k^- & 1 - \beta_k^- \end{bmatrix} \begin{bmatrix} e^{u_{k-1} S_{k-1}} & 0 \\ 0 & e^{v_{k-1} S_{k-1}} \end{bmatrix} \begin{bmatrix} \overline{A}_{k-1} \\ \overline{B}_{k-1} \end{bmatrix} \quad (5)$$

Then set $\overline{F}_m = 0$ and $\overline{G}_m = e^{v_m Z_m}$, and start another computation at $k = m - 1$ for $n \le k \le m - 1$ with:

$$\begin{bmatrix} \overline{F}_k \\ \overline{G}_k \end{bmatrix} = \begin{bmatrix} \alpha_k^+ & \beta_k^+ \\ 1 - \alpha_k^+ & 1 - \beta_k^+ \end{bmatrix} \begin{bmatrix} e^{-u_{k+1} S_{k+1}} & 0 \\ 0 & e^{-v_{k+1} S_{k+1}} \end{bmatrix} \begin{bmatrix} \overline{F}_{k+1} \\ \overline{G}_{k+1} \end{bmatrix} \quad (6)$$

This completes all terms in $E[T]$. The proof is omitted.

Special cases If the last segment $m = n$ includes the starting point $z = D$ then:

$$E[T] = \frac{r + \mu}{r\mu} [\sqrt{\frac{b_n^2 + 2c_n(\lambda_n + r)}{b_1^2 + 2c_1(\lambda_1 + r)}} \overline{A}_n e^{u_n(D - Z_{n-1})} - 1] \quad (7)$$

and if the search space is homogenous $m = n = 1$ then [15]:

$$E[T] = \left(\frac{1}{r} + \frac{1}{\mu}\right) \left[e^{u_1 D} - 1\right]$$

If the searcher consumes energy only when it is moving, and not while it is is the lost or while it is waiting to be retransmitted, then the average energy consumed is simply given by

$$E[J] = (1 + E[T]) \sum_{k=1}^{m} \int_{Z_{k-1}}^{Z_k} f_k(z) dz. \quad (8)$$

Search in a Protected Neighbourhood Consider the case where the neighbourhood of the object being sought, up to a distance S, is protected by randomly located traps that destroy the searcher. In the rest of the search space accidental destruction of the searcher may occur, but at much lower rate. Thus we take $m = n = 2$, so that $E[T]$ is obtained from (7) with $\lambda_2 = \lambda$ and $\lambda_1 = \lambda + \gamma$, $\gamma > 0$. Figure 1 shows the manner in which $E[T]$ sharply increases with γ, for S ranging between 10 and 15, $D = 100$, $b_2 = b_1 = 0.25$, $c_1 = c_2 = 1$, $\lambda = 0$. Also $\mu = 1/10$ and r is set to the value that minimises $E[T]$ when $\gamma = 0$ and $S = 10$. Figure 2 indicates that S and λ can be chosen to maximise the protection offered to the object being sought.

Phase Transition when Defence is More Effective than Search The destruction of the searcher and the time-out, both relaunch the search and allow the searcher to improve its chances to attain the object. However we will see that if the object being sought is heavily defended when the searcher gets close, then the searcher may never attain it. In Figure 3 we observe that if $\log \lambda_k = \frac{1}{k\rho}$, as ρ becomes very small, $E[T]$, and also $E[J]$ in (8) which is not shown on the graph, tend to infinity despite the fact that the search speed and its accuracy grow as the searcher approaches the object. Thus if the searcher's speed of approach to the object grows faster than the rate at which the searcher is destroyed then both $E[T]$ and $E[J]$ remain finite or tend to zero, while in the opposite case they tend to infinity, presenting a form of phase transition.

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