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Abstract	This paper studies interconnected wireless sensors with the paradigm of Energy Packet Networks (EPN) which were previously introduced. In the EPN model, both data transmissions and the flow of energy are discretized, so that an energy packet (EP) is the minimum amount of energy (say in microjules) that is needed to process and transmit a data packet (DP) or to process a job. Previous work has modeled such systems to determine the relation between energy flow and DP transmission, or to study the balance between energy and the processing of jobs in Cloud Servers. The lack of energy, in addition to processing times, is the main source of latency in networks of sensor nodes. Thus this paper models this phenomenon, and shows that under some reasonable conditions, assuming feedforward flow of data packets and local consumption and leakage of energy, such networks have product form solutions.

Interconnected Wireless Sensors with Energy Harvesting

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Abstract. This paper studies interconnected wireless sensors with the paradigm of Energy Packet Networks (EPN) which were previously introduced. In the EPN model, both data transmissions and the flow of energy are discretized, so that an energy packet (EP) is the minimum amount of energy (say in microjules) that is needed to process and transmit a data packet (DP) or to process a job. Previous work has modeled such systems to determine the relation between energy flow and DP transmission, or to study the balance between energy and the processing of jobs in Cloud Servers. The lack of energy, in addition to processing times, is the main source of latency in networks of sensor nodes. Thus this paper models this phenomenon, and shows that under some reasonable conditions, assuming feedforward flow of data packets and local consumption and leakage of energy, such networks have product form solutions.

1 Introduction and Previous Work

Information and communication technologies (ICT) steadily increase their energy consumption by about 4% per year [22] reaching roughly 5% of the worldwide electrical energy consumption in 2012 [18], but there is also hope that ICT can also reduce the energy consumption in other areas such as transportation for daily commutes [17,26]. However the users of ICT use more complex multimedia technologies [14] which are ever more demanding in energy because of their computational complexity and communication bandwidth, so that progress will be needed to reduce limit this growth through more efficient microelectronics and new technologies such as energy harvesting for computation and communications [25,23,21,1,24,16]. Thus recent research has addressed new technologies based on energy harvesting and so as to minimise the non-renewable energy consumption for given communication tasks [12,13].

Furthermore, earlier work [15] has also shown that smart routing [7] based on QoS can be also used to reduce overall energy consumption in a network.

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Because of the random nature of both data flows and of energy, in such contexts it become convenient to view not just data and computation but also energy itself in discrete units. This has given rise to the energy packet (EP) paradigm [9,8] based on mathematical models such as G-Networks [6]. Such discrete representations are useful also to capture the stochastic nature of compute-communications, energy harvesting and data sensing in interconnected micro-electronic and computer-communication systems.

Contribution. In this paper we pursue a modeling approach developed in [10,11] where energy harvesting wireless sensors are modeled, assuming that data collection times and the time needed to harvest significant amount of energy, is substantially higher than the time needed to transmit a packet when energy is available. Thus the stochastic system representation that is used assumes finite and positive data and energy arrival rates to nodes and zero service or data transmission times when energy is available. In this paper this approach is developed for a sensor network that contains two nodes and packets travel in feedforward mode through two nodes, or just through one node, before successfully exiting the network. The structure we consider also includes not just energy harvesting, but also the realistic situation when energy leakage may occur. We prove that, under mild conditions, the equilibrium distribution of the continuous time Markov chain underlying the model has product-form solution and hence the derivation of the performance indices can be carried out efficiently. Indeed, many models of wireless sensor networks suffer the problem of the state-space explosion (see, e.g., [20,5,4]) that makes the exact analysis of the underlying stochastic process very difficult especially for large networks. Product-form allows us to derive the performance measures of the WSN by the analysis of each sensor *as if* it were isolated. From a theoretical point of view, although the model can be seen as belonging to the wide class of G-networks [6], the product-form is new and depends on some conditions on the model's rates that will be discussed later.

Structure of the Paper. The paper is structured as follows. In Section 2 we introduce the mathematical model of a single sensor. Section 3 describes the model of interconnected sensors and proves the product-form equilibrium distribution from which we derive some mean performance indices such as the expected number of data packets enqueued in a sensor and its throughput. Finally, Section 4 concludes the paper.

2 The Mathematical Model

In this section we consider a single wireless sensor which operates with energy harvesting. We assume that as soon as the sensor has both a data packet to transmit and enough energy to transmit that packet, the transmission takes place very rapidly so that it may be represented as a "zero time" or instantaneous transmission. We denote each device by the acronym EHWS (Energy Harvesting

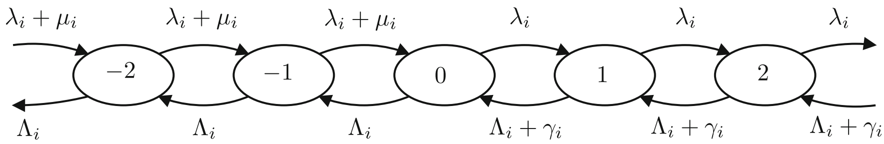

Fig. 1. CTMC underlying the model of sensor i

Wireless Sensor). The EHWS has an unlimited buffer for data packets and an unlimited "buffer" or battery for energy "packets" since we represent energy in discrete units. Thus an *energy packet* (EP) is the exact amount of energy required to transmit a data packet. Just as data packets are assumed to be collected into the EHWS in discrete packets of data, we consider that the harvested energy is also collected into the device's storage battery in discrete units (the energy packets).

Thus for an EHWS i, the state can be represented by an integer n_i, where i identifies the EHWS, where $n_i = 0$ means that the device has neither energy nor data packets, while $n_i > 0$ means that it currently stores n_i data packets but no energy, while $n_i < 0$ means that it stores n_i energy packets but no data packets. We also suppose that the EHWS harvests energy packets at a rate Λ_i while it collects data packets at a rate λ_i. Also, each device looses energy through leakage at rate μ_i, and we will assume that packets themselves will be discarded with a time-out represented by a rate γ_i.

If one considers a single EHWS whose state is represented by the integer n_1, it becomes clear that it may be modelled as a random walk, provided all the rates are parameters of independent exponentially distributed random variables, and that this random walk is ergodic provided that $\Lambda_1 + \gamma_1 > \lambda_1$ and $\lambda_1 + \mu_1 > \Lambda_1$ as in Figure 1. As a consequence of the exponential assumption, DPs and EPs arrive at the node according to independent and homogeneous Poisson processes.

Furthermore, we can readily see that its stationary distribution for an isolated node i is given by:

$$\pi_i(n_i) = \pi_i(0)\left(\frac{\lambda_i}{\Lambda_i + \gamma_i}\right)^{n_i} \qquad \text{if } \; n_i > 0\,,$$

$$\pi_i(n_i) = \pi_i(0)\left(\frac{\Lambda_i}{\lambda_i + \mu_i}\right)^{-n_i} \qquad \text{if } \; n_i < 0\,,$$

$$\pi_i(0) = \left[1 + \frac{\frac{\lambda_i}{\Lambda_i+\gamma_i}}{1 - \frac{\lambda_i}{\Lambda_i+\gamma_i}} + \frac{\frac{\Lambda_i}{\lambda_i+\mu_i}}{1 - \frac{\Lambda_i}{\lambda_i+\mu_i}}\right]^{-1} .$$

3 Interconnected Sensor Nodes

In this section we study the steady-state behaviour of a network of EHWSs. We assume the topology of the network to be such that each DP is forwarded at most once by a node to another node before leaving the system. We prove that the CTMC underlying such a network of sensors has a product-form equilibrium distribution under some conditions on the energy leakage rate and on the DP time-outs. In this setting, in a network of several EHWS, the transmission of a data packet (DP) from any one of the sensors may either result in the data packet arriving at the second sensor, or it may be directed towards the "exit" so that it is removed from the network. Therefore, a DP may visit at most two nodes, and that this occurs in one of four ways:

- A DP arrives from outside the network (e.g. through sensing) at one of the sensors; if that EHWS has no energy, then it is placed in the DP buffer.
- The DP arrives from outside the network at sensor i that does have energy; it consumes energy and is transmitted, then it leaves the network with probability p_{i0}.
- The DP arrives from outside the network at sensor i that does have energy; it consumes energy and is transmitted, and then arrives at sensor j with probability p_{ij}, $j \neq i$. If sensor j *does not* have energy, the DP stays there.
- Finally the DP arrives from outside the network to sensor i that does have energy; it consumes energy and is transmitted with probability p_{ij} to sensor $j \neq i$; if sensor j *does* have energy, the DP leaves the network.

A network whose topology satisfies the conditions of the two-steps routing is shown in Figure 2.

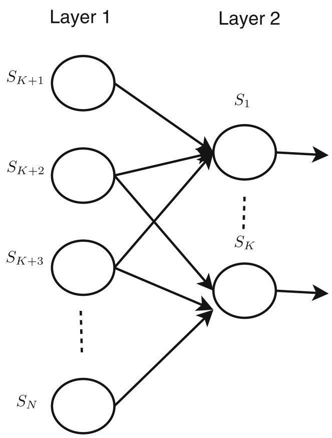

Fig. 2. A network of EHWSs with two layers. In this case packets are forwarded by at most one node.

3.1 Product-Form Analysis

In order to derive the product-form equilibrium distribution for a network of EHWSs we assume that the rate associated with the time-out (energy leakage)

Table 1. Table of notation for sensor i

Λ_i	Arrival rate of EPs at sensor i
λ_i	Arrival rate of DPs at sensor i from outside
γ_i	Time-out rate for DPs in state $n_i > 1$
γ_i^0	Time-out rate for DPs in state $n_i = 1$
μ_i	Leakage rate for EPs in state $n_i < -1$
μ_i^0	Leakage rate for EPs in state $n_i = -1$
p_{ij}	Probability of routing from EHWS i to EHWS j
$n_i > 0$	Number of DPs buffered by the sensor
$n_i < 0$	Number of EPs stored by the sensor battery

in states $n_i > 1$ ($n_i < -1$) may be different from that in state $n_i = 1$ ($n_i = -1$). We consider a network of N EHWSs in which a DP is forwarded at most once. Let us consider a network of N nodes whose state is $\mathbf{n} = (n_1, \ldots, n_N)$ with $n_i \in \mathbb{N}$. Formally, let $\mathbf{P} = (p_{ij})$ be the routing matrix with $1 \leq i, j \leq N$ and let p_{i0} be the probability that a DP leaves the network after visiting node i, i.e., $\sum_{n=1}^{N} p_{in} + p_{i0} = 1$. We recall that each sensor is described by the set of parameters shown in Table 1.

The transition rates of the CTMC underlying the network of EHWSNs are $q(\mathbf{n}, \mathbf{n}') =$

- $\Lambda_i p_{ij}$ if $\mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ and $n_i > 0$ (transmission of a DP from node i to j due to the harvesting of an EP)
- $\lambda_i p_{ij}$ if $\mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ and $n_i < 0$ (transmission of a DP from node i to node j due to the availability of a new data)
- Λ_i if $\mathbf{n}' = \mathbf{n} - \mathbf{e}_i$ and $n_i \leq 0$ (harvesting of a new EP from node i)
- $\Lambda_i p_{i0} + \gamma_i^0$ if $\mathbf{n}' = \mathbf{n} - \mathbf{e}_i$ and $n_i = 1$ (transmission of a DP to the outside or timeout of a DP at node i)
- $\Lambda_i p_{i0} + \gamma_i$ if $\mathbf{n}' = \mathbf{n} - \mathbf{e}_i$ and $n_i > 1$ (transmission of a DP to the outside or timeout of a DP at node i)
- λ_i if $\mathbf{n}' = \mathbf{n} + \mathbf{e}_i$ (generation of a new DP at node i)
- $\lambda_i p_{i0} + \mu_i^0$ if $n_i = -1$ (transmission of a DP from node i to the outside when EP are available or leakage of an EP)
- $\lambda_i p_{i0} + \mu_i$ if $n_i < -1$ (transmission of a DP from node i to the outside when EP are available or leakage of an EP)

We define for each EHWS the following quantity v_i.

$$v_i = \lambda_i + \sum_{n=1}^{N} \frac{\Lambda_n \lambda_n}{\Lambda_n + \gamma_n} p_{ni} .$$

We will show that v_i denotes the total arrival rate of DPs at EHWS i.

The following assumption on the time-out and energy leakage rates will be sufficient to prove the product-form stationary distribution of the EHWS network.

Assumption 1. We assume the following relations on the timeout settings:

- $v_i + \mu_i = \Lambda_i + \gamma_i$
- $\mu_i^0 = v_i + 2\mu_i$
- $\gamma_i^0 = \Lambda_i + 2\gamma_i$

Theorem 1. *Under the constraints of Assumption 1, given two interconnected EHWS j and k, their joint equilibrium distribution has the following product-form:*

$$\pi(n_j, n_k) = G g_j(n_j) g_k(n_k)\,, \tag{1}$$

where

$$g_i(n_i) = \begin{cases} 1 & \text{if } n_i = 0 \\ \frac{v_i}{\Lambda_i+\gamma_i^0}\left(\frac{v_i}{\Lambda_i+\gamma_i}\right)^{n_i-1} & \text{if } n_i \geq 1 \\ \frac{\Lambda_i}{v_i+\mu_i^0}\left(\frac{\Lambda_i}{v_i+\mu_i}\right)^{-n_i-1} & \text{if } n_i \leq -1 \end{cases} \tag{2}$$

with $i = j, k$ and G is the normalising constant:

$$G = \left(1 + \frac{\frac{v_j}{\Lambda_j+\gamma_j^0}}{1-\frac{v_j}{\Lambda_j+\gamma_j}} + \frac{\frac{\Lambda_j}{v_j+\mu_j^0}}{1-\frac{\Lambda_j}{v_j+\mu_j}}\right)^{-1} \left(1 + \frac{\frac{v_k}{\Lambda_k+\gamma_k^0}}{1-\frac{v_k}{\Lambda_k+\gamma_k}} + \frac{\frac{\Lambda_k}{v_k+\mu_k^0}}{1-\frac{\Lambda_k}{v_k+\mu_k}}\right)^{-1}.$$

The proof is given in appendix.

Remark 1 (On the conditions for the product-form). It is worth of notice that the product-form expression given by Theorem 1 is subject to the constraints on the rates stated in Assumption 1. The fact that some product-form results have conditions which depend on the model's transition rates is not new (see e.g., the conditions on the service rate in the First Come First Service queues of the BCMP theorem [3], or the product-forms derived in [2,19]). Nevertheless, in this case the conditions required by Assumption 1 are not strict since they give a relation on the time-out setting. Informally, we require that the sum of the energy harvesting rate and the packet time-out rate (consumption rate of packets) must be equal to the sum of the DPs arrival rate and the energy leakage rate (consumption rate of energy packets). This balance basically states that we must consume the DPs with the same rate at which we consume the EPs and can be reached by opportunely setting the time-out rate of DPs. Similar considerations hold for the time-out taking the model to state 0.

The product-form expression given by Theorem 1 can be used to compute some important performance indices such as a EHWS throughput, energy efficiency and expected number of enqueued DPs.

Proposition 1. *In stability, the DP throughput of EHWS i is:*

$$TH_i = \frac{\Lambda_i v_i}{\mu_i + v_i}\,. \tag{3}$$

The proof follows from the simplification of the expression:

$$\sum_{n=-\infty} G_i g_i(n) v_i + \sum_{n=1}^{\infty} G_i g_i(n) \Lambda_i \,,$$

where G_i is the normalising constant for $g_i(n)$. The total power consumption of the EHWS is clearly Λ EPs for unit of time. However, not all this energy is used to transmit DPs because some will suffer the energy leakage. The efficiency of the EHWS is given by the ratio of the EPs used for transmitting data and the total EPs consumed.

Proposition 2. *In stability, the efficiency of EHWS i is:*

$$\eta_i = 1 - \frac{\mu_i}{\gamma_i + \Lambda_i} \,. \tag{4}$$

Indeed, the rate of consumption of EPs for DPs transmission is given by:

$$G_i \left[\sum_{n=-\infty}^{-1} g_i(n) v_i + \sum_{n=1}^{\infty} g_i(n) \Lambda_i \right] .$$

Finally, we can derive the expected number of DPs in the queue.

Proposition 3. *In stability, the expected number of DPs in the queue of EHWS i is:*

$$N_i = \frac{\gamma_i \lambda_i}{\mu_i(\gamma_i + \mu_i)} \,.$$

The expression of N_i can be derived by the simplification of the sum:

$$G_i \sum_{n=1}^{\infty} n g_i(n) \,.$$

4 Conclusion

This paper has shown that a plausible and novel model of two interconnected energy harvesting wireless sensors with discretised energy harvesting and storage, and a feedforward data packet communication pattern has product form solution in its state that represents both the amount of energy and the data packet backlog at each sensor. In future work we expect that these results can be generalised to other topologies and to arbitrarily large networks.

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A Proof of Theorem 1

Since the network structure has a topology in which a packet if forwarded by at most one EHWS, we can prove the theorem by just considering a tandem of two sensors in the state space (n_1, n_2). Equation (1) of Theorem 1 can be rewritten as:

$$\pi(n_1, n_2) = G g_1(n_1) g_2(n_2)$$

and

$$v_1 = \lambda_1 , \quad v_2 = \lambda_2 + \frac{\Lambda_1 \lambda_1 p_{12}}{\Lambda_1 + \gamma_1} .$$

For the sake of simplicity we give the proof for $p_{12} = 1$ and $\lambda_2 = 0$. The proof proceeds by substitution in the system of global balance equations of the underlying CTMC. Let us consider the case in which $n_1 > 0$. Then, the corresponding balance equation of a state (n_1, n_2), with $n_2 \in \mathbf{Z}$ is:

$$\begin{aligned}
&\pi(n_1, n_2)\big(\lambda_1 + \Lambda_1 + \gamma_1^0 \delta_{n_1=1} + \gamma_1 \delta_{n_1>1} + \Lambda_2 + \gamma_2^0 \delta_{n_2=1} \\
&\qquad\qquad + \gamma_2 \delta_{n_2>1} + \mu_2^0 \delta_{n_2=-1} + \mu_2 \delta_{n_2<-1}\big) \\
&= \underbrace{\pi(n_1+1, n_2)\gamma_1}_{A} + \underbrace{\pi(n_1, n_2+1)\left(\Lambda_2 + \gamma_2 \delta_{n_2>0} + \gamma_2^0 \delta_{n_2=0}\right)}_{B} + \underbrace{\pi(n_1-1, n_2)\lambda_1}_{C} \\
&\qquad + \underbrace{\pi(n_1+1, n_2-1)\Lambda_1}_{D} + \underbrace{\pi(n_1, n_2-1)(\mu_2 \delta_{n_2\le -1} + \mu_2^0 \delta_{n_2=0})}_{E}
\end{aligned}$$

We divide the RHS by $\pi(n_1, n_2)$. We have:

$$\frac{A}{\pi(n_1, n_2)} = \frac{\lambda_1 \gamma_1}{\Lambda_1 + \gamma_1} \tag{5}$$

For part B:

$$\frac{B}{\pi(n_1,n_2)} = \frac{\lambda_1 \Lambda_1}{\Lambda_1+\gamma_1}\frac{1}{\Lambda_2+\gamma_2\delta_{n_2>0}+\gamma_2^0\delta_{n_2=0}} \cdot \left(\Lambda_2+\gamma_2\delta_{n_2>0}+\gamma_2^0\delta_{n_2=0}\right)\delta_{n_2\geq 0} + \left(\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2\delta_{n_2<-1}+\mu_2^0\delta_{n_2=-1}\right)\frac{1}{\Lambda_2}\Lambda_2\delta_{n_2<0}$$

$$= \frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2\delta_{n_2<-1}+\mu_2^0\delta_{n_2=-1} \tag{6}$$

For part C:

$$\frac{C}{\pi(n_1,n_2)} = \frac{\Lambda_1+\gamma_1\delta_{n_1>1}+\gamma_1^0\delta_{n_1=1}}{\lambda_1}\lambda_1 = \Lambda_1+\gamma_1\delta_{n_1>1}+\gamma_1^0\delta_{n_1=1} \tag{7}$$

For part D:

$$\frac{D}{\pi(n_1,n_2)} = \frac{\lambda_1}{\Lambda_1+\gamma_1}\left(\frac{\Lambda_2}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0}}\right)\Lambda_1\delta_{n_2\leq 0} + \frac{\lambda_1}{\Lambda_1+\gamma_1}\left((\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1})\frac{\Lambda_1+\gamma_1}{\Lambda_1\lambda_1}\right)\Lambda_1\delta_{n_2>0}$$

which simplifies to:

$$\frac{D}{\pi(n_1,n_2)} = \frac{\lambda_1\Lambda_2\Lambda_1}{\Lambda_1\lambda_1+(\Lambda_1+\gamma_1)(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0}}\delta_{n_2\leq 0}+\Lambda_2\delta_{n_2>0} + \gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1} \tag{8}$$

For part E:

$$\frac{E}{\pi(n_1,n_2)} = \frac{\Lambda_2}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0}}\left(\mu_2\delta_{n_2\leq -1}+\mu_0\delta_{n_2=0}\right)$$

$$= \frac{\Lambda_2(\Lambda_1+\gamma_1)}{\Lambda_1\lambda_1+(\Lambda_1+\gamma_1)(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0})}\left(\mu_2\delta_{n_2\leq -1}+\mu_0\delta_{n_2=0}\right) \tag{9}$$

Summing Equations (8) and (9) we have:

$$\frac{D+E}{\pi(n_1,n_2)}$$

$$= \frac{\lambda_1\Lambda_2\Lambda_1+\Lambda_2\Lambda_1(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0})+\Lambda_2\gamma_1(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0})}{\Lambda_1\lambda_1+(\Lambda_1+\gamma_1)(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0})}\delta_{n_2\leq 0} + \Lambda_2\delta_{n_2>0}+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}$$

$$= \frac{\lambda_1\Lambda_2\Lambda_1+\Lambda_2(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0})(\Lambda_1+\gamma_1)}{\Lambda_1\lambda_1+(\Lambda_1+\gamma_1)(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0})}\delta_{n_2\leq 0}+\Lambda_2\delta_{n_2>0}+\gamma_2^0\delta_{n_2=1} + \gamma_2\delta_{n_2>1} = \Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1} \tag{10}$$

Finally, we sum Equations (5), (6), (7), (10) and obtain:

$$\begin{aligned}&\frac{\lambda_1\gamma_1}{\Lambda_1+\gamma_1}+\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2\delta_{n_2<-1}+\mu_2^0\delta_{n_2=-1}\\&\qquad+\Lambda_1+\gamma_1\delta_{n_1>1}+\gamma_1^0\delta_{n_1=1}+\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}\\&=\lambda_1+\mu_2\delta_{n_2<-1}+\mu_2^0\delta_{n_2=-1}+\Lambda_1+\gamma_1\delta_{n_1>1}+\gamma_1^0\delta_{n_1=1}+\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}\end{aligned}$$

which is exactly the LHS of Equation (1) divided by $\pi(n_1,n_2)$, as required.

We now consider the case $n_1=0$. The balance equations for states $(n,0)$ are:

$$\begin{aligned}&\pi(0,n_2)(\lambda_1+\Lambda_1+\Lambda_2+\gamma_2\delta_{n_2>1}+\gamma_2^0\delta_{n_2=1}+\mu_2\delta_{n_2<-1}+\mu_2^0\delta_{n_2=-1})\\&\quad=\underbrace{\pi(1,n_2-1)\Lambda_1}_{A}+\underbrace{\pi(-1,n_2-1)\lambda_1}_{B}+\underbrace{\pi(-1,n_2)\mu_1^0}_{C}+\underbrace{\pi(1,n_2)\gamma_1^0}_{D}\\&+\underbrace{\pi(0,n_2+1)\left(\Lambda_2+\gamma_2\delta_{n_2\geq1}+\gamma_2\delta_{n_2=0}\right)}_{E}+\underbrace{\pi(0,n_2-1)\left(\mu_2\delta_{n_2\leq-1}+\mu_2^0\delta_{n_2=0}\right)}_{F}\end{aligned}\tag{11}$$

Let us compute $(A+B)/\pi(0,n_2)$:

$$\begin{aligned}&\frac{\lambda_1}{\Lambda_1+\gamma_1^0}\left(\frac{\Lambda_2}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0}}\right)\Lambda_1\delta_{n_2\leq0}\\&\qquad+\frac{\lambda_1}{\Lambda_1+\gamma_1^0}\left(\frac{\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}}{\frac{\lambda_1\Lambda_1}{\Lambda_1+\gamma_1}}\right)\Lambda_1\delta_{n_2>0}\\&\qquad+\frac{\Lambda_1}{\lambda_1+\mu_1^0}\left(\frac{\Lambda_2}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0}}\right)\lambda_1\delta_{n_2\leq0}\\&\qquad+\frac{\Lambda_1}{\lambda_1+\mu_1^0}\left(\frac{\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}}{\frac{\lambda_1\Lambda_1}{\Lambda_1+\gamma_1}}\right)\lambda_1\delta_{n_2>0}\\&=\left(\frac{\lambda_1\Lambda_1}{\Lambda_1+\gamma_1^0}+\frac{\lambda_1\Lambda_1}{\lambda_1+\mu_1^0}\right)\left(\frac{\Lambda_2}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0}}\delta_{n_2\leq0}\right.\\&\qquad\left.+\frac{\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}}{\frac{\lambda_1\Lambda_1}{\Lambda_1+\gamma_1}}\delta_{n_2>0}\right)\end{aligned}$$

Notice that by Assumption 1 we have

$$\Lambda_1+\gamma_1^0=2(\Lambda_1+\gamma_1)=2(\lambda_1+\mu_1)=\lambda_1+\mu_1^0\,.\tag{12}$$

This allows us to rewrite the first term of the product as $\lambda_1\Lambda_1/(\Lambda_1+\gamma_1)$ and hence simplify the expression as follows:

$$\begin{aligned}\frac{A+B}{\pi(0,n_2}&=\frac{\lambda_1\Lambda_1\Lambda_2}{\Lambda_1\lambda_1+(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2\leq-1})(\Lambda_1+\gamma_1)}\delta_{n_2\leq0}\\&\qquad+\Lambda_2\delta_{n_2>0}+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}\end{aligned}\tag{13}$$

We now compute $(C+D)/\pi(0,n_2)$ by using Relation (12):

$$\frac{C+D}{\pi(0,n_2)} = \frac{\Lambda_1}{\lambda_1+\mu_1^0}\mu_1^0 + \frac{\lambda_1}{\Lambda_1+\gamma_1^0} = \frac{\Lambda_1(\lambda_1+2\mu_1)+\lambda_1(\Lambda_1+2\gamma_1)}{2(\Lambda_1+\gamma_1)}$$
$$= \frac{\lambda_1\Lambda_1+\mu_1\Lambda_1+\lambda_1\gamma_1}{\Lambda_1+\gamma_1} = \lambda_1 + \frac{\mu_1\Lambda_1}{\lambda_e+\gamma_1} \quad (14)$$

Let us derive $E/\pi(0,n_2)$:

$$\frac{E}{\pi(0,n_2)} = \frac{\frac{\lambda_1\Lambda_1}{\Lambda_1+\gamma_1}+\mu_0\delta_{n_2=-1}+\mu\delta_{n_2<-1}}{\Lambda_2}\Lambda_2\delta_{n_2<0}$$
$$+\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}\frac{1}{\Lambda_2+\gamma_2^0\delta_{n_2=0}+\gamma_2\delta_{n_2\geq 1}}(\Lambda_2+\gamma_2^0\delta_{n_2=0}+\gamma_2\delta_{n_2\geq 1})\delta_{n_2>0}$$
$$= \frac{\lambda_1\Lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=-1}+\mu_2\delta_{n_<-1} \quad (15)$$

Notice that the sum $(C+D+E)/\pi(0,n_2) = \lambda_1+\Lambda_1+\mu_2^0\delta_{n_2=-1}+\mu_2\delta_{n_<-1}$. We now compute $(A+B+F)/\pi(0,n_2)$ to obtain the remaining terms of the LHS of Equation (11) divided by $\pi(0,n_2)$. By using Equation (13):

$$\frac{F}{\pi(0,n_2)}+\frac{A+B}{\pi(0,n_2)} = \frac{\Lambda_2}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2\leq -1}}\left(\mu_2\delta_{n_2\leq -1}+\mu_2^0\delta_{n_2=0}\right)$$
$$+\frac{\lambda_1\Lambda_1\Lambda_2}{\Lambda_1\lambda_1+(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2\leq -1})(\Lambda_1+\gamma_1)}\delta_{n_2\leq 0}+\Lambda_2\delta_{n_2>0}+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}$$
$$= \lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}\,.$$

The last case is when $n_1<0$. In this case the GBE associated with states (n_1,n_2) have the form:

$$\pi(n_1,n_2)\big(\lambda_1+\Lambda_1+\mu_1^0\delta_{n_1=-1}+\mu_1\delta_{n_1<-1}+\Lambda_2+\mu_2\delta_{n_2<-1}+\mu_2^0\delta_{n_2=-1}$$
$$+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}\big)$$
$$= \underbrace{\pi(n_1-1,n_2-1)\lambda_1}_{A}+\underbrace{\pi(n_1-1,n_2)\mu_1}_{B}+\underbrace{\pi(n_1+1,n_2)\Lambda_1}_{C}$$
$$+\underbrace{\pi(n_1,n_2-1)(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2\leq -1})}_{D}+\underbrace{\pi(n_1,n_2+1)(\Lambda_2+\gamma_2^0\delta_{n_2=0}+\gamma_2\delta_{n_2\geq 1})}_{E} \quad (16)$$

We divide the equation by $\pi(n_1,n_2)$ and consider the RHS:

$$\frac{A}{\pi(n_1,n_2)} = \frac{\Lambda_1}{\lambda_1+\mu_1}\left(\frac{\Lambda_2\delta_{n_2\leq 0}}{\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0}}\right.$$
$$\left.+\frac{\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1}}{\frac{\Lambda_1\lambda}{\Lambda_1+\gamma_1}}\delta_{n_2\geq 1}\right)\lambda_1$$

$$= \frac{\Lambda_1}{\lambda_1+\mu_1}\frac{\Lambda_2(\Lambda_1+\gamma_1)\lambda_1\delta_{n_2\le 0}}{\Lambda_1\lambda_1+(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0})(\Lambda_1+\gamma_1)} + \frac{\Lambda_1}{\lambda_1+\mu_1}\frac{(\Lambda_1+\gamma_1)(\Lambda_2+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1})\lambda_1\delta_{n_2\ge 1}}{\Lambda_1\lambda_1}$$

Recalling that by Assumptions 1 we have $\lambda_1+\mu_1=\Lambda_1+\gamma_1$ this simplifies to:

$$\frac{A}{\pi(n_1,n_2)} = \frac{\Lambda_2\lambda_1\Lambda_1\delta_{n_2\le 0}}{\Lambda_1\lambda_1+(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0})(\Lambda_1+\gamma_1)} + \Lambda_2\delta_{n_2\ge 1}+\gamma_2^0\delta_{n_2=1}+\gamma_2\delta_{n_2>1} \quad (17)$$

$$\frac{B+E}{\pi(n_1,n_2)} = \frac{\Lambda_1\mu_1}{\lambda_1+\mu_1} + \frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}\frac{\Lambda_2+\gamma_2^0\delta_{n_2=0}+\gamma_2\delta_{n_2\ge 1}}{\Lambda_2+\gamma_2^0\delta_{n_2=0}+\gamma_2\delta_{n_2\ge 1}}\delta_{n_2\ge 0} + \left(\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=-1}+\mu_2\delta_{n_2<-1}\right)\frac{1}{\Lambda_2}\Lambda_2\delta_{n_2<0}$$

$$\frac{\Lambda_1\mu_1}{\lambda_1+\mu_1}+\frac{\Lambda_1\lambda_1}{\Lambda_1+\gamma_1}+\mu_2^0\delta_{n_2=-1}+\mu_2\delta_{n_2<-1} = \Lambda_1+\mu_2^0\delta_{n_2=-1}+\mu_2\delta_{n_2<-1} \quad (18)$$

$$\frac{C}{\pi(n_1,n_2)} = \frac{\lambda_1+\mu_1\delta_{n_1<-1}+\mu_1^0\delta_{n_1=-1}}{\Lambda_1}\Lambda_1 = \lambda_1+\mu_1\delta_{n_1<-1}+\mu_1^0\delta_{n_1=-1} \quad (19)$$

$$\frac{D}{\pi(n_1,n_2)} = \frac{\Lambda_2(\Lambda_1+\gamma_1)(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2\le -1})\delta_{n_2\le 0}}{\Lambda_1\lambda_1+(\mu_2^0\delta_{n_2=0}+\mu_2\delta_{n_2<0})(\Lambda_1+\gamma_1)} \quad (20)$$

The analysis of the global balance equation system is concluded by observing that summing Equations (17), (18), (19), (20), we obtain the LHS of the balance equation (16) as required.

As regards the derivation of the normalising constant it is sufficient to compute the sum of the geometric series given by summing Equation (2) over the state space $(-\infty,+\infty)$. $\square$